Graphics for Engineers, Architects, and Builders:

A MANUAL FOR DESIGNERS, AND A TEXT-BOOK FOR SCIENTIFIC SCHOOLS.

TRUSSES AND ARCHES

ANALYZED AND DISCUSSED BY GRAPHICAL METHODS.

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IN THREE PARTS.

I.

ROOF-TRUSSES: Diagrams for Steady Load, Snow, and Wind.

II.

BRIDGE-TRUSSES: Single, Continuous, and Draw Spans; Single and Multiple Systems; Straight and Inclined Chords.

III.

ARCHES, in Wood, Iron, and Stone, for Roofs, Bridges, and Wall-Openings; Arched Ribs and Braced Arches; Stresses from Wind, and Change of Temperature.

PART II.

BRIDGE-TRUSSES.

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GENERAL PREFACE.

THERE is not the necessity which would have existed a few years ago to write a few words explanatory of graphical methods of analysis, and to urge their accuracy, convenience, and adaptability to all types of structure. If, however, some readers now glance at such subjects for the first time, it will be sufficient to point out, that, as forces may be represented by straight lines of definite magnitude and direction, the same operations which are performed by mathematical analysis may be more easily carried out by geometrical construction upon the drawing-board; and, as the whole system is founded upon the parallelogram of forces, the results deduced by a brief chain of reasoning are theoretically accurate, and only depend for their numerical exactness upon the scale of the diagrams, and the care with which they are drawn. Any draughtsman who gives the method a fair trial, and then takes the trouble to compare the results with those obtained by mathematical formulæ, will be surprised to see how closely the two agree. The difference is much less than can be taken account of in designing the proportions of the several parts of the structure. The simplest tools alone are needed, — drawing-board, T-square, triangles, and scale.

This work is intended for office use by any designer who may have occasion to prepare plans for a structure intended to support a load above any opening, — a roof, or bridge-truss, or an arch, — of any type, span, and distribution of load. It is also hoped that these three parts may be found serviceable as a text-book for engineering and architectural students, and may aid in clearing away some of the obscurity which is apt to surround certain parts of the subjects treated in the following pages, especially those of continuous bridges and arches. From his own experience as an engineer and as an instructor, the author has been led to go more into detail than an ordinary expansion of the methods would require, and in some cases to repeat what has been explained earlier; for he has found that both the practical man, taking up the subjects at intervals as required for use, and the student, finding so much that is new, need a certain amount of repetition to fix the principles clearly in mind. It will be found that students generally grasp readily, and evince a strong liking for, graphical methods. Mathematical and graphical analysis go well hand in hand, the latter aiding the student to more clearly apprehend the meaning of the former. Indeed, the mathematical formulæ may be, if desired, deduced directly from the diagrams. If sufficient time is allowed between the class-room exercises for the construction in the drawing-room of problems suggested by the text, the results accruing from this study will be most satisfactory.

Some special remarks in regard to subjects treated will be found prefacing the several parts.

C. E. G.

PREFACE TO PART II.

THE general method of analysis called "The Method by Area Moments," which is the foundation of the following pages, was first printed in 1874 in "Graphical Method for the Analysis of Bridge-Trusses," &c., having been discovered and taught by the author the year preceding. Part I., discussing roof-trusses, was founded upon what is known as Professor Clerk-Maxwell's method of drawing diagrams, and has little which is original, except the carrying-out of details. It is thought that Part II. will be found to contain much which is new, and, it is hoped, valuable. That this part is not a second edition, and simply a reprint, of the book above referred to, will be seen, when it is noticed that there is more than twice the amount of text, and sixty-three in place of fifteen illustrative figures. Chapter III. is devoted to an analysis of various types of trusses with horizontal chords having single systems of bracing, and is almost all new. The treatment of multiple systems of bracing, comprising Chapter IV., is entirely new. The construction for the bowstring girder, and the chapter on deflection of beams, are here introduced for the first time: partially continuous trusses and pivot-draws, with turn-table tipping or stable, have also been added.

As it is now four years since this method of area moments for the analysis of continuous girders was first given to the public by the author, and as no statement that such a method can be found in any other place has appeared, the author feels warranted in putting forth a claim for priority of discovery and originality. The attention of the reader is asked to the extreme simplicity and the generality of the formulæ here deduced for pier moments in continuous girders and draw-spans, and to the facility and brevity with which all the usual formulæ of the text-books, simple or intricate, for the slope and deflection of beams of one or more spans, are derived without the use of the integral calculus, as usually understood. These results have never been obtained, to our knowledge, in this way before. If the reader knows the position of the centre of gravity of a triangle, he need here accept nothing on faith. Following this investigation, it will be seen that the *Three Moment Theorem* in its special and general forms is deduced directly from the equation of *Area Moments* by simple substitution. The truth of all these formulæ is now for the first time made clear to the reader who has no knowledge of the higher mathematics. Indeed, it may not be amiss to state that all the propositions here advanced may be understood by one who possesses a very moderate knowledge of mathematics and mechanics.

The endeavor has been made to take nothing for granted: hence graphical proof has been offered in regard to the extent of load which produces maximum bending moment and maximum shear, in regard to the effect of inclination of chords on web-stresses, in regard to absence of stress in the braces of the parabolic girder, &c. Some minor points will be found to possess novelty: prominent among these is the construction for

web-stresses in the bowstring, parabolic truss, which will be found to be extremely simple in application. Whether the special constructions for trusses with multiple systems of bracing will be useful or not, depends upon the frequency with which they are needed: some of them are thought to be elegant; they are all short in construction. The method here set forth of drawing the line which limits the maximum shear ordinates in a single span truss, and which was given in the former edition without proof, is here shown by a mathematical demonstration to be theoretically exact.[1] A simple and convenient diagram for the effect of a locomotive, or a load of greater than average intensity, is inserted. Very many of the details are thought to be new, and a comparison is invited for accuracy and brevity between these constructions and those found in other books.

The author would also ask a candid comparison of his method of Area Moments with the German method, as a means of solving problems in continuous girders, believing that, for the small number of operations, and hence simplicity and accuracy, as well as comprehensiveness, it will be found at least equally convenient.

The attempt has been made, by the use of capitals for the lettering of the trusses, accented capitals for the moment diagrams, small letters for the shear diagrams, and numerals for the load line, &c., to render a reference to the several diagrams of each figure easy.

Part III., on arches of all types, is in an advanced stage of preparation, about one-half of it having been once already in

[1] This demonstration in its essential features, as here given, was the work of Mr. Charles A. Marshall, at the time a student in the University of Michigan.

type, and will follow this part during the coming season. It is hoped that it will be found equally simple and intelligible with the previous parts.

C. E. G

ANN ARBOR, MICH., July 5, 1878.

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CHAPTER I.

INTRODUCTION.

1. **Triangle of Forces.** — We know, from one of the fundamental theorems of mechanics, that, if three forces not parallel act upon a body and are in equilibrium, their directions must intersect at a common point, and that these forces must be proportional to the sides of a triangle drawn parallel to their directions; and that, conversely, if three forces are parallel to the three sides of any triangle, and proportional in magnitude to the same sides, with the direction of the several forces taken in the order obtained by passing over the sides of the triangle in succession, these three forces applied at one point must be in equilibrium.

If the weight W, Fig. 1, be hung from two points, A and B, by the cords A C and C B, we may find the pull or tension on each of the cords by drawing a vertical line *a b*, equal, by any convenient scale, to the given weight W, and then drawing the lines *a c* and *b c*, from the extremities of *a b*, parallel to A C and B C, intersecting at *c*. Then will *a c* and *b c* represent, by the same scale by which W was laid off, the pull on the cords A C and B C. The arrows on the cords represent the directions of the stresses relatively to the point C, and the arrows on the triangle *a b c* must follow one another in order round the triangle.

If W and the two forces *b c* and *c a* had been given, we might have reversed the problem, and have found the direction and length of two cords, which, while supporting W, would have exerted the given tensions on the points A and B. As it is possible to draw any number of triangles on the line *a b*, we may have a like number of arrangements of cords, from A and B, to carry W.

2. **Equilibrium Polygon; Stress Diagram.**—If we have several weights suspended from the points A and B, Fig. 2, by means of a cord whose weight is either neglected, or considered as included in the given weights, we may find the tensions in the several portions of the cord by successive applications of the above process. The weight at C is balanced by the tensions of A C and C D, and a triangle of forces may be drawn for the point C. Next, taking the weight W_2, and the tension, just found, in C D, we may draw a triangle for the point D.

But since, when we draw a triangle for each loaded point, each portion of the cord will be represented by a side in two of the triangles, and since all of the triangles will have one vertical side, they may be brought together into one figure by the following construction: Draw a vertical line 1–2, and lay off on the same, $W_1 = 1–3$, $W_2 = 3–4$, $W_3 = 4–5$, and $W_4 = 5–2$. Draw 1–0 parallel to A C, and 2–0 parallel to B F; connect the point of intersection 0 with the other points of division on the vertical line: then will 1–0, 3–0, 4–0, 5–0, and 2–0 be equal to the stresses in A C, C D, D E, E F, and F B; and the portions of the cord must be parallel to the lines radiating from 0, or otherwise the cord will not be in equilibrium.

The figure 0 1 2 is styled by many authors the *force polygon;* but we prefer to call it the *stress diagram*, from its analogy to the stress diagram of a frame. The vertical line 1–2 will be called the *load line;* and the cord from A to B, which hangs in what is known as a *funicular polygon*, we shall have occasion hereafter to designate as the *equilibrium* or *moment polygon.* The point 0 is called the *pole* of the stress diagram.

3. **Equilibrium Polygon for Bridge-Trusses.** — If the weights, their horizontal distances from the point A, and the horizontal distance of B from A, were alone given, we might draw an *equilibrium polygon* to satisfy these conditions by assuming the pole 0 in any convenient position, drawing the radiating lines to the several points on 1–2, then drawing a line parallel to 1–0, from A, to meet the vertical through W_1, — from the point C, so found, drawing a line parallel to 3–0 to meet a vertical through W_2, — and so on; the last line parallel to 2–0 determining the position of B, since its horizontal distance from A was previously given. This latter simple application of the method for finding an equilibrium polygon will suffice for the investigation of bridge-trusses which is to follow.

4. **Shearing Force.** — If all of the forces which act on a structure lie in one vertical plane, they may be resolved into horizontal and vertical components in that plane. All the external forces considered in this part are vertical; viz., the weight of the structure, the imposed load, and the reactions of the points of support. If a beam or truss supported in any manner, as in Fig. 4 or 6, is cut by an imaginary vertical plane, we shall have on one side of the plane of section the resultant of all the external vertical forces which act upon the portion of the beams or truss upon that side; and, on the other side of the section, another resultant of the external forces upon the second side: since there is perfect equilibrium, these two resultants must be equal and opposite to one another. As it is proved in mechanics that a force at any distance from a given point is equivalent, so far as its action in regard to that point is concerned, to the same force at the point and a moment equal to the product of the force by its perpendicular distance from the point, the two resultants above will, in the first place, give the two equal and opposite forces at the section, represented by the arrows in the panel A C D B of Fig. 3. These forces tend to move the two sections of the beam or truss in opposite directions; and there must be a resisting stress in the fibres of the beam at the section, or a brace in the panel of the truss, to

prevent the movement. From the nature of the movement, this stress is called a *shearing stress;* and, as it must be equal to the resultant of vertical forces on *one side* of the plane of section, this resultant is usually styled the *shearing force* or shear at the section: it is positive on one side of the section, and negative on the other.

A reference to the panel A C D B of Fig. 3 will, we think, enable the reader to see how the direction of the shear determines the kind of stress in the bracing or web members. At the section in this panel the shear is supposed to be upward on the left side of the section denoted by the dotted line, and downward on the right side. If we imagine the horizontal members or *chords* to be severed, it is evident that a strut in the diagonal C B, drawn in the sketch, or a tie in the diagonal A D of the rectangle, will be needed to prevent one part of the truss from falling. Or if we suppose the points of intersection of the chords with the web members to be joints free to turn, as is the case with pin-connected parts, it is clear, that, if no diagonal member exists, the rectangle A′ C′ D′ B′ will be distorted into a parallelogram, one diagonal shortening, and the other lengthening. This movement also shows that the chords alone, if jointed, have no power to resist shear, or convey a vertical force, but that a strut must be introduced in the diagonal which tends to shorten, or a tie in that one which tends to lengthen, if the distortion and falling of the truss are to be prevented. *The web members*, therefore, *resist the shearing forces.*

If, in the same or any other panel, the shear is in the opposite direction, a strut in the diagonal A D, or a tie in the diagonal C B, will be necessary. The arrows in the panels of the truss E F are supposed to represent the direction of the shear on the *left side* of a section in the successive panels. The diagonals for the given shears are struts, and change in direction as soon as the shear is reversed.

5. **Bending Moment.** — Further, as indicated in the last section, the same external vertical forces on one side of the plane

of section have *moments* about any point of the section; and the resultant moment will be obtained by multiplying the resultant previously referred to by the horizontal distance of its point of application from the section, or by multiplying each external force by its horizontal distance, or lever arm, from the plane of section, adding those products which have a tendency to produce rotation in one direction, and subtracting the sum of those products which tend to produce rotation in a contrary direction. The remainder is called the *bending moment* at the section, and would cause rotation of the structure about the section in the direction shown by the larger sum, if it were not exactly balanced, as required by the condition that equilibrium exists, by an equal and opposite moment on the other side of the section. These two equal moments neutralize one another by means of the *moment of resistance* in the given vertical section of the beam or truss, arising from the resistances to extension and compression of the fibres or pieces in the two portions, upper and lower, of the beam or truss; each stress being multiplied by its lever arm from any convenient point in the section. These horizontal resisting stresses are due to the attempt of the external forces to bend the beam.

Since the external forces are vertical, their only effects at the plane of section must be those due to a vertical force and a moment, called, as stated, the *shear* and the *bending moment:* and the opposing resistances in the material at the section will be, first, a vertical shearing stress; and, second, horizontal tensions and compressions. Thus we have at any section the horizontal and vertical components of the stresses in the parts cut by the section, and are able to derive from them the direct stresses in the pieces. It will then be convenient to determine the shearing forces and bending moments at all points of the structure as a preliminary step. If, from any consideration, we know the stress in one or more pieces at a section, the remainder of the horizontal and vertical components must exist in the remaining pieces. As the external forces which are here treated are vertical, the horizontal components of the stresses at any section must balance among themselves.

6. Example of Equilibrium Polygon for a Beam; Supporting Forces. — Suppose that a beam, such as is represented in Fig. 4, is supported at the two ends, and has four unequal weights situated, as in the sketch, at unequal intervals upon it. Let the weight of the beam be included in the imposed weights. It is required to find the *supporting forces*, or reactions, at the abutments A and B, and the shearing force and bending moment at all points. Draw a stress diagram, as before, by laying off on a vertical line 1–2, Fig. 4, the weights W_1, W_2, &c. Assume the pole 0 at a convenient distance from 1–2, and draw radiating lines from 0 to all points of division on 1–2. Draw vertical lines through the points of support and the loaded points A, C, D, E, F, and B. Commence at a convenient point, A′, in the vertical through A, and draw A′ C′ parallel to 0–1. From the point C′, where this line cuts the vertical from the first weight, draw C′ D′ parallel to 0–3. Continue the same process until the last line F′ B′, drawn parallel to 0–2, meets the vertical line let fall from B, the second point of support.

If A′ C′ D′ E′ F′ B′ were a cord, fastened at A′ and B′, and under the action of the given weights in their given positions, it would be in equilibrium, as already shown. The pull on A′ would be in the direction A′ C′ and of the amount 0–1. The pull on B′ would be in the direction B′ F′ and of the amount 2–0. If the two ends of the cord, in place of being fastened at A′ and B′, were attached to the two ends of a rigid bar, A′ B′, the whole system might be suspended by two cords from the points A and B without disturbing the equilibrium of the polygon; for, if we draw 0–5 in the stress diagram parallel to B′ A′, we can see that the inclined pulls at A′ and B′ are, by the introduction of the bar A′ B′, each decomposed into a thrust along the bar and a force acting vertically downwards, as shown by the arrows. At the point A′ we have three forces, which, in the stress diagram, must make the triangle whose sides are 0–1, 1–5, and 5–0. In the same way, for the point B′, we have the sides of the triangle 0–5, 5–2, and 2–0. The thrust 5–0 at A′ balances the thrust 0–5 at B′; and there remain 1–5 and 5–2, or

P_1 and P_2, the forces exerted by this system on the points of support A and B. Any other system of framing, loaded and supported in the same way, must give the same pressure on the points of support; and thus the upward reactions of the abutments for the given loaded beam will be determined.

Therefore, to find the supporting forces: Having drawn an equilibrium polygon, connect A′ with B′, draw 0–5 through 0, parallel to this closing line; and the two portions into which the load line is thus divided will be the forces required.

7. **Shear Diagram.** — As the shearing force at any section is the resultant of the vertical forces on one side of the plane of section, the shearing force at any point between A and C will be P_1, or 1–5, the only vertical force on the left of C, and acting upwards on the left side of the section. Between C and D the shearing force will be $P_1 - W_1$, or 3–5; between D and E it will be $P_1 - W_1 - W_2$ or 4–5; and so on, being that part of load included by the lines of the stress diagram parallel to the two lines of the moment polygon cut by a vertical section.

But if we draw a horizontal line $a\,b$ equal to A B; then at a lay off $a\,g$ equal to P_1, and upwards, as denoting the direction of P_1; next draw from g a line $g\,i$ parallel to $a\,b$; then at the point i, vertically over c and under C, measure off $i\,k$ equal to W_1, downwards in the direction of action of the weight; draw $k\,l$ horizontally, then $l\,m$ downwards equal to W_2, and finally $m\,n\,p\,q\,r\,s$, — we shall have a broken line, the ordinate to which, from any point of $a\,b$, will be the shear on the *left* of the section at the corresponding point of A B. When the W's which have been subtracted exceed P_1, the broken line passes below $a\,b$; and finally, on arriving at f, having subtracted all the W's, we have a shearing force equal but opposite to P_2. So, at any section, the shear at one side of the plane of section, obtained by the subtraction just described, will always be equal and opposite to the shear on the other side of the plane of section, obtained by working from B, with the supporting force P_2 as the minuend. Such a result is required to fulfil the condition of equilibrium.

If the load is continuous, in place of being concentrated at a certain number of points, the successive ordinates will vary in length, diminishing with the amount of load passed; so that the bounding line will not be horizontal in the loaded portions. If the load is of uniform intensity, the upper extremities of the ordinates will be bounded by a straight inclined line.

8. **Moment Diagram.** — Lastly, to find graphically the bending moment at any point. Take the point S. The bending moment on the beam at S is, when we take the moments of external forces on the left of S,

$$P_1 \cdot AS - W_1 \cdot CS - W_2 \cdot DS.$$

Drop a perpendicular from S, cutting the polygon A′ E′ B′ at I and K. Produce C′ D′ and A′ C′ to meet this perpendicular at L and M. Also draw C′ N horizontally. In the stress diagram draw a horizontal line, 0–6, through 0, to meet the load line. Call this line H. It is the horizontal projection of the stress in each side of the equilibrium polygon, which projection is well known to be *constant* for a system of vertical loads. The triangles C′ N M and 0 6 1, having their sides parallel, are similar, and we have the proportion,

$$MN : C'N = 6\text{–}1 : 0\text{–}6.$$

From the similar triangles C′ N L and 0 6 3 we have

$$LN : C'N = 6\text{–}3 : 0\text{–}6:$$

hence

$$\frac{MN - LN}{C'N} = \frac{(6\text{–}1) - (6\text{–}3)}{0\text{–}6},$$

or

$$ML \cdot (0\text{–}6) = (3\text{–}1) \cdot C'N.$$

But

$$0\text{-}6 = H;\ 3\text{–}1 = W_1:$$

therefore

$$H \cdot ML = W_1 \cdot C'N = W_1 \cdot CS.$$

In the same way

$$H \cdot LK = W_2 \cdot DS, \text{ and } H \cdot IM = P_1 \cdot AS:$$

therefore the bending moment

$$M = H\,(IM - ML - LK) = H \cdot IK.$$

Hence the bending moment at any point of the beam is *proportional* to the ordinate, from $A'B'$ to the equilibrium polygon, vertically below that point, and is equal to the product of that ordinate by H, the constant horizontal component of the tensions in the polygon.

The ordinate must be measured by the scale to which the beam is drawn; and the line which represents H, by the scale to which the load line is measured off: but the two figures may be of the same scale, that is, number of tons and feet to the inch, or of different scales, whichever is more convenient. If the pole 0 be taken at such a distance from the load line that 0–6 shall measure ten or one hundred units of weight in length, the bending moment will be readily obtained by scaling the ordinate, and moving the decimal point one or two places to the right, as the case may be.

9. **Relation between Shear and Moment Diagrams.** — It will be noticed that the line which limits the ordinates for shear crosses the horizontal line, or, in other words, the shear changes sign, at that point of the beam where the ordinate to the equilibrium polygon, and hence the bending moment, is a maximum. This relation between the shear and moment diagrams always exists; for, since 0–5 was drawn parallel to the closing line $A'B'$ of the equilibrium polygon, the maximum ordinate will be found at E', where the two sides $D'E'$ and $E'F'$, which make the angle E', are parallel to lines from 0 on opposite sides of 0–5; and at that place sufficient weights will have been subtracted from the reaction 1–5 to reduce it to zero, and cause the shear line to pass the horizontal line at *e*.

10. **Moment due to any Force; Centre of Gravity of Weights.** — It has been proved in § 8, that the portion LM of the ordinate at S, intercepted between the lines $C'M$ and $C'L$, is proportional to the moment of W_1 about S; that LK, cut off by $D'L$ and $D'K$, is proportional to the moment of the weight which is over D'; and that IM, included between the lines from A', is proportional to the moment of P_1 about S: hence it follows that the particular portion of the bending moment

at any point of a beam which is due to one or more of the external forces is readily ascertained by dropping a vertical from the point in question, and prolonging those two sides of the equilibrium polygon which include between them the given forces. The portion of the vertical line thus cut off, when multiplied by H, will be the desired quantity. This property of the equilibrium polygon will be of service later.

If A′ C′ and B′ F′ are prolonged to an intersection at R, and the rest of the polygon is removed, the two sides A′ R and B′ R will be in equilibrium, and will undergo the tensions 0–1 and 0–2, if the entire load 1–2 is concentrated at R. Since the two reactions will be unchanged, a vertical line drawn through R gives the position of the resultant of the applied weights of Fig. 4. Similarly, if any two sides of the equilibrium polygon are prolonged until they meet, the resultant of the included weights will lie in the vertical through that intersection. Thus the resultant of W_1, W_2, and W_3, must pass through T. This operation may also be termed finding the position horizontally of the centre of gravity of the weights.

11. **Maximum Bending Moment at a Section.** — If the beam A B, of Fig. 5, carries at first only the weight W at C, the bending moments at all points of the beam will be proportional to the ordinates from A B to A C′ B, and all of the moments will be of the same kind, tending to make the beam concave on the upper side: such moments we prefer to call positive, since they are the ones with which we are most familiar. In the stress diagram, 3–0 will be the accompanying value of H, and 2–3 the supporting force at A. In the same manner, an equal W at D gives the triangle A D′ B, with 0′ 2 1 in the stress diagram. The other two equal W's give the remaining parts of the diagram. If all the values of H are equal, as they have been made here, the several ordinates below each point of the beam may be added arithmetically, since the bending moments are alike in kind, producing the figure A″ C″ E″ B″, whose ordinates will be multiplied by 0–3, or any other equal line representing H, to produce the bending moments. Exactly

the same figure would be obtained by laying off the whole load on a vertical line, taking the pole at the above distance from the vertical, and proceeding as usual.

It will be evident, from the preceding steps, that, as every additional weight increases the bending moments at all points of a beam or truss of one span supported at both ends, *the greatest possible bending moments at all points*, in case such a structure is subject to a moving load, *will be found when the span is entirely covered with the rolling load.*

12. **To bring the Closing Line Horizontal.** — Since the closing line $A'' B''$ is parallel to the line from the pole to that point on the load line which divides it into the two supporting forces or reactions, if it is desired that A'' and B'', the extremities of the funicular polygon, shall fall on the same horizontal line as in this figure, it is necessary to divide the load line into the two reactions, and then to assume the pole 0 in the same horizontal line with the point of division. It is generally of no consequence that $A'' B''$ should be horizontal: the equilib rium polygon has the same properties in any position.

13. **Maximum Shearing Force at a Section.** — Returning again to the case where the weight W alone acted upon the beam of Fig. 5 at C, we know that the shear diagram will be represented by $a\,c\,g\,h\,i\,b$, where $a\,c = P_1 = 2\text{-}3$, and $g\,h = W$, which is subtracted at the point vertically below C. On the left of any section between A and C the shear is positive or upward, and equal to P_1; while at any point between C and B the shear is negative on the left of a section, and equal to $-P_2$. In the same way, when W alone is placed upon the beam at D, we obtain the shear diagram $a\,d\,k\,l\,m\,b$, in which the shear again changes in sign at the weight. The other two shear diagrams belong to the remaining weights. If, then, it is possible to have some or all of these weights on the beam at one time, it is evident that the maximum positive shear on the left of a section, at all points between A and C, will be found when all of the weights are placed upon the beam; for we shall then have a shear whose value is the sum of the separate positive ordi-

nates of the four shear diagrams just drawn. For any section between C and D it is manifest that the positive shear will be greatest when W is removed from C. For a section between D and E the two weights on the right will give positive shears, and anywhere between E and F a positive shear on the left of a section can only be obtained from the weight at F. It is also plain that the four weights together will make the maximum negative shear on the left of a section between B and F, and that the weights must be removed in succession as we pass them to find the greatest negative shears at different points.

Hence follows the rule, that *the greatest positive or upward shear on one side of any section will exist*, for a beam or truss of one span supported at both ends, *when all possible moving load is placed upon the other segment only of the span;* and the greatest negative shear, when the moving load covers the segment on the same side.

14. **Diagram of Maximum Shear.** — These maximum positive values of shear may be grouped in one figure, when they will produce the diagram $a' c' d' e' f' b'$; but it must be remembered that *the shears represented by the ordinates of this diagram are not co-existent*, but occur in succession as the loads are added from one end. They are, however, useful, since they give the greatest shears, which must be guarded against in the structure. If the reader will draw a shear diagram for a complete load, and then for loads over a less extent of span, he can readily compare the diagrams with the last one of Fig. 5, and see the difference, and at the same time the agreement of the *maximum values* at successive sections.

If the load is uniformly distributed, the broken line $c' d' e' f' b'$ will become a continuous curve, which can be proved to be parabolic; and, if a simple method of constructing the diagram is given, the maximum shear at each point for such a load can be readily found.

CHAPTER II.

SINGLE-SPAN TRUSSES WITH HORIZONTAL CHORDS. — GENERAL TREATMENT.

15. **Equilibrium Polygon applied to a Bridge; Data.** — Let the method, as thus far developed, be applied to a bridge-truss with parallel chords, under a moving load of given intensity. The truss is represented by A B C D, Fig. 6, supported at A and D. In order to deal with moderate dimensions, we will suppose that the span is 80 feet, height of truss 10 feet, fixed load $\frac{1}{4}$ ton per running foot, moving load $\frac{1}{2}$ ton per running foot, each for one truss. The scales of the figure and diagrams are, as shown below them, 30 feet = 1 inch, and 30 tons = 1 inch. The small weights represent the fixed load arising from the truss and platform, as if concentrated on the joints of the lower chord; the larger weights represent the moving load. As each panel is 10 feet long, the load from the bridge will be $2\frac{1}{2}$ tons at each joint, and from the rolling load 5 tons. The points A and D will each carry one-half of a panel weight: these weights will cause no stress in the truss, and might be neglected altogether; but it will be found convenient to plot them on the load line, as thus the total weight of the truss will be accounted for, and the shearing force will be more readily obtained.

16. **Polygon for a Partial Load.** — Suppose that the rolling load extends from the abutment D to the joint G inclusive. Draw a vertical line, and lay off 1–2 equal to $\frac{1}{2}$ $(2\frac{1}{2} + 5)$ tons, the weight at D; next 2–3 = $7\frac{1}{2}$ tons at L, 3–4 = $7\frac{1}{2}$ tons at K, and so on to 6–7 at G; then 7–11 = $2\frac{1}{2}$ tons at F, 11–12 = $2\frac{1}{2}$

tons at E, and finally 1¼ tons at A, reaching a point 13, mid way between 12 and 14. Next assume the pole 0, which is so taken here that H measures 20 tons. Leave out of consideration 1–2, the weight at D; and starting at D′, a convenient distance vertically below D, draw D′ L′, L′ K′, . . . E″ A″ parallel respectively to 2–0, 3–0, 4–0, &c.; the last line, E″ A″, being drawn parallel to 12–0. The equilibrium polygon will be completed by drawing the closing line from A″ to D′.

17. Shear Diagram for the Same Load. — Draw in the stress diagram 0–9 parallel to A″ D′: 13–9 and 9–1 will be the supporting forces at A and D. The shearing force at any section will be represented by the ordinates to a broken line, constructed similarly to *g i k l m*, &c., of Fig. 4; the ordinate at *a* being equal to 12–9. If, however, the concentrated loads of this truss are supposed to be distributed over the horizontal line A D with an intensity of one-fourth ton per foot from A to a point half way between F and G, and of three-fourths ton per foot for the remaining distance, the total load will be unchanged, the reactions will be the same; but the ordinate *a g* will equal 13–9, *d n* will equal —(9–1), and the ordinates at all points will be limited by *g p n*. The line *g p* must have a declivity or slope corresponding to the intensity of ¼ ton per foot, or 2½ tons per panel; while *p n* inclines at three times that rate. The angle *p* occurs in the middle of a panel, where the intensity of the load changes. The difference between the ordinate at *a* to the inclined line *g p* and the one which would have been plotted for the broken line, or between 13–9 and 12–9, is the half-panel weight directly supported at A. Then as, in reaching the middle of successive panels, we shall have passed beyond just as much distributed load as the amount of concentrated load which is here carried at the joints, the broken line which falls by steps, and the inclined lines *g p n* of the figure, will intersect one another in the middle of each panel. A small portion of the broken line is sketched near *g*, and, for another case, near *i*: hence, as it is easier to draw *g p n* than the broken line, we may find the shear in each panel by measuring the ordinate to *g p n* in the *middle of each panel.*

18. **Effect on Diagrams of Movement of Load.**— If the rolling load retires, so that J is the last fully loaded point, we shall have three loads of $7\frac{1}{2}$ tons each, and the points from E to I will carry only $2\frac{1}{2}$ tons each. We may make available a considerable portion of the polygon already drawn. Since 4–5 represents the load on J, we can lay off the smaller loads below; the one on E falling at 6–16, and the half load at A reaching a little beyond 16. Constructing a second equilibrium polygon as we did the first, we shall get $D' K' I' G'' F'' A'''$; and, drawing 0–8 parallel to $A''' D'$, we get the new supporting forces for this position of the rolling load: hence we find $j r$ and $r i$ as we did $g p$ and $p n$. Do not forget that $d i = 1\text{–}8$, and not 2–8, and similarly for $a j$; for the inclined lines always start from the end of an ordinate which represents the entire abutment reaction, including the weight directly supported there.

If the load extends over the whole truss, our load line will be 1–10, and the equilibrium polygon will be $A' E' F' G' I' \ldots D'$. As the pole 0 was taken opposite the middle of 1–10, and as, for an entire uniform load, the two supporting forces are equal, the line $A' D'$ is horizontal. The shear diagram for this case is $a b v c d$.

19. **Maximum Moments and Shears.**— Without carrying out in detail the construction for every possible position of the moving load, we see indications which will render needless the drawing of so many diagrams. First, as pointed out in § 11, the bending moment at any point of the truss is greatest when the whole bridge is loaded. The polygon $A' G' J' D'$ will give ordinates of the greatest length, and these ordinates multiplied by H give the bending moments. Second, as in § 13, the positive shearing force at any point is greatest, when, of the two portions into which the point divides the span, that segment is fully loaded which lies on the opposite side of the plane of section to the shear considered. For example, the shear in the panel F G, when the rolling load extends from D to G, is $h p$. If the rolling load is less, or covers a less extent, the supporting force at A will be less than $a g$, and the line parallel to

g p will therefore pass below *p*. On the other hand, if the moving load advances farther, the supporting force at D will be greater than *d n*, and the line parallel to *n p* will again pass below *p*. The ordinate *h p* is, therefore, equal to the greatest possible positive shear in the panel F G.

20. **Curve for Maximum Shears.** — It is evident that the greatest value of the reaction at either abutment will be one-half of the weight of truss and full load, and that the least value will be one-half of the weight of the unloaded truss. In the first case the shear diagram will be *a b v c d;* and, in the second case, *a l v k d; a l* and *a b* being respectively the half weight of truss and the half weight of truss and load, and the two inclined lines cutting *a d* at the middle of the span. From an inspection of the two diagrams already drawn, whose angles fall at *p* and *r*, it will be seen, that for loads coming on at D, and extending gradually across the bridge, the vertex *r* of the angle of the shear diagram will, for successive advances of load, take a series of positions from *m* in the last panel, on the line *l k*, through *r* and *p*, to *t* in the first panel, on the line *b c*, when the rolling load has finally reached E. If, then, we know the path described from *t* to *m*, and can construct it, all the maximum ordinates can be readily obtained. If the rolling load were applied uniformly foot by foot to a beam, the desired locus would be a *parabola*, extending from *b* to *k*, described on *b v* and *v k* as tangents; but as our locus is to extend from *t* to *m*, for loads concentrated at intervals, the lines *b v* and *v k* will not be tangents. The construction will now be shown: the proof of its truth will follow.

Draw a horizontal line A B, Fig. 7, equal to the span of the truss. At one end of it, as A, erect a perpendicular A C, equal to one-half weight of truss fully loaded, to the scale of load line used; and at B draw on the opposite or negative side of A B a perpendicular B D, equal to one-half weight of truss alone. From the extremities of these lines draw C O and D O to the middle point O of A B. Divide C O and D O each into the same number of equal parts as there are panels in the truss.

Number the points of division in the same direction on each. beginning at C and O with 0. Draw straight lines 1–1, 2–2, &c., between the points having the same numbers. Then the vertical ordinates from A B to the intersections of consecutive lines in the series (which ordinates will come at the middle of the panels of A B, if that line be divided into panels to match the truss) will be the *exact shears* required in the panels in which they occur for loads concentrated at joints as here taken, and already described in Fig. 6.

As verticals dropped from the panel points of the truss will divide the lines C O and D O, or $b\,v$ and $v\,k$ of Fig. 6, into half the desired number of parts, it is only necessary to bisect these parts to obtain the desired points. The reader can complete this figure on a large scale, and notice the intersections of the several lines between C O and D O. Any consecutive lines, such as 3 and 4, will intersect vertically above the number found by adding their numbers, in this case 7, always an odd number; and the intersection thus occurs exactly in the middle of a panel.

21. **Proof; General Formula for Shear.** — First, to deduce a formula for the shear in any panel. Let

N = number of panels in the truss; l = length of truss in feet;

$\frac{l}{N}$ = length of one panel; W = fixed load at a joint;

W' = moving load at a joint; n = number of any joint in the bottom chord from the left abutment, the abutment being numbered 0;

F_n = shear in the nth panel, between the nth and $(n+1)$ st joints.

If the joints from $n+1$ to $N-1$ inclusive are each loaded with $W+W'$, and the others with W only, the shear from steady load in the nth panel will be the supporting force $\frac{1}{2}(N-1)\,W$ less the loads, $n\,W$, between the left abutment and the section, or $[\frac{1}{2}(N-1)-n]\,W$. The amount of rolling load on the truss is $(N-1-n)\,W'$; and the shear from this load at the section in the nth panel will be the same as the supporting force at A, which is equal to the above rolling load multiplied by $\frac{N-n}{2}\cdot\frac{l}{N}$, the distance of its centre of gravity from the opposite abutment, and divided by the span l, or $(N-1-n)\,\frac{N-n}{2\,N}\,W'$. Therefore

$$F_n = \tfrac{1}{2}(N-2n-1)\,W + (N-n-1)\,\frac{N-n}{2\,N}\,W'.$$

22. **Equation of the Shear Line.** — Second, let us take the origin of co-ordinates in Fig. 7 at O, A B being the axis of abscissas, and the ordinates being measured vertically. Let v and m be the co-ordinates of any point of division P, v being vertical, and measured to the scale by which A C and B D were laid off, and m being horizontal, expressed in units of a half-panel length. The equation of the line O C is

$$v = -\tfrac{1}{2} m \, (W + W'):$$

the equation of the line O D is

$$v = -\tfrac{1}{2} \, m \, W.$$

Let P′ P″ be any line of the series previously described. The co-ordinates of the point P″ are

$$m'', \text{ and } v'' = -\tfrac{1}{2} \, m'' \, W.$$

The co-ordinates of the point P′ are

$$m' = m'' - N, \text{ and } v' = -\tfrac{1}{2} \, m' \, (W + W') = -\tfrac{1}{2} \, (m'' - N) \, (W + W').$$

From analytic geometry we have the general equation of a line in terms of co-ordinates of two points in it,

$$v - v' = \frac{v' - v''}{m' - m''} \, (m - m').$$

Substituting the value of the other co-ordinates in terms of m'', as given above, we obtain

$$v + \tfrac{1}{2} (m'' - N) \, (W + W') = \frac{-\tfrac{1}{2} (m'' - N) \, (W + W') + \tfrac{1}{2} m'' \, W}{m'' - N - m''} \, (m - m'' + N)$$

$$= -\tfrac{1}{2} \, (m - m'' + N) \, (W + W') + m'' \, (m - m'' + N) \, \frac{W'}{2 \, N},$$

or
$$v = -\tfrac{1}{2} \, m \, (W + W') + m'' \, (m - m'' + N) \, \frac{W'}{2 \, N}, \qquad (a)$$

which is the equation of the line P′ P″, in terms of m''.

Writing in this equation $m'' + 1$ for m'', we shall obtain the equation of the consecutive line in the series, which is

$$v = -\tfrac{1}{2} \, m \, (W + W') + (m'' + 1) \, (m - m'' - 1 + N) \, \frac{W'}{2 \, N}$$

$$= -\tfrac{1}{2} \, m \, (W + W') + (m'' + 1) \, (m - m'' + N) \, \frac{W'}{2 \, N} - (m'' + 1) \, \frac{W'}{2 \, N}. \qquad (b)$$

If we eliminate m'' from equations (a) and (b), we shall have the equation of the locus of the intersection of the consecutive lines. To do so, subtract (a) from (b), obtaining

$$0 = (m - m'' + N) \, \frac{W'}{2 \, N} - (m'' + 1) \, \frac{W'}{2 \, N}; \text{ or}$$

$$0 = m - 2 \, m'' + N - 1:$$

whence
$$m'' = \tfrac{1}{2} \, (N + m - 1).$$

Substitute this value in (a), and

$$v = -\tfrac{1}{2} m (W + W') + \tfrac{1}{2} (N + m - 1) [m - \tfrac{1}{2} (N + m - 1) + N] \frac{W'}{2N}$$

$$= -\tfrac{1}{2} m (W + W') + \tfrac{1}{4} (N + m - 1)(N + m + 1) \frac{W'}{2N}, \quad (c)$$

which is the equation of the locus required.

The points of division at the ends of $P' P''$ are both numbered m'', or the horizontal distance of one end from O to the right is the same as the horizontal distance of the other end from C to the right; that is, the projection of the moving line is a constant quantity.

The value of m is shown above to be

$$m = -N + 2m'' + 1.$$

It will be remembered that m is the abscissa, measured in half-panel lengths, of the point of intersection of the two lines whose extremities are at m'' and $m'' + 1$: hence, to find where the ordinate to this point comes on the axis of abscissas, measure from A, which is $-N$ half-panel lengths from O, $2m'' + 1$ half-panel lengths. Putting this fact in general terms, we may say that the ordinate to the intersection of the nth line with the $(n + 1)$st line of the series comes at a distance of $2n + 1$ half-panel lengths to the right from A, or in the middle of the nth panel.

To find the value of this ordinate in terms of n, we write in (c) $m'' = n$, and $m = -N + 2n + 1$ thus producing

$$v = \tfrac{1}{2} (N - 2n - 1)(W + W') + n (-N + 2n + 1 - n + N) \frac{W}{2N}$$

$$= \tfrac{1}{2} (N - 2n - 1) W + (N - n - 1) \frac{N - n}{2N} W',$$

which was the value of the shear F deduced in § 21.

23. **Diagonals and their Stresses.** — The shearing force in any panel is then obtained, for example, in the panel E F, Fig. 6, by erecting the ordinate $e\,s$ to the *shear curve* (if we may so style it) at the middle e of the panel, the curve having been constructed by the method of Fig. 7. As the panel would change from its rectangular form in case the diagonal were removed, as already explained in § 4, the diagonal S F alone must resist the shear, and the diagonal will be a tie. The ordinate $e\,s$ being the vertical component of the tension in this tie, we need only draw $s\,f$ parallel to S F, from s, until it meets the horizontal line, when $s\,f$ will be the tension in S F, according to the scale of loads. It is well to bear in mind that a horizon-

tal member can transmit no vertical force, unless it acts as a beam, which it can only do when it has a force or load imposed directly upon it; but a direct force of tension or compression applied at one end of a piece passes through without change to the next joint.

A similar line from p will give the tension in R G, and so on to r. The point r, under the middle of the panel I J, will be the last one where we shall find any positive or upward shear on the *left* of the plane of section, which occurs when the load extends from D to J. As shear of an opposite kind calls for a diagonal, if a tie, sloping in the other direction, in the panel I J will be found the last necessary tie parallel to B E.

The remaining panels from J to D would have, therefore, the diagonals represented in the sketch; but a rolling load coming on at A, and extending towards D, would cause shears similar to those given by ordinates between r and m, but greater in amount than those stresses. The diagram would be completed by drawing on $l\ v$ and $v\ c$ another shear curve similar to the one previously constructed; but, as the figure would be exactly the reverse of $b\ t\ s\ p\ r\ m\ k$, it is sufficient to find the stresses from A to J, and then make the truss from D to G symmetrical with A J. The heavier the rolling load compared with the fixed load, the farther will the diagonals sloping one way pass beyond the middle of the span. Those diagonals which are between the centre and the abutment to which they convey their load are usually called *main* braces: those beyond the centre are termed *counterbraces*, or simply *counters*. Fig. 6 has but two counters. Some designers, for constructive reasons. place counters in many or all of the remaining panels.

The occurrence of two diagonals in one panel should lead to no ambiguity as to which one is under strain. They must both be tension, or both compression members; and the points developed in § 4 will enable one to see which diagonal will act for a given shear. As one diagonal tends to shorten, the brace occupying that place, if a compression member, will resist the tendency; or, if a tie, will spring or buckle under a slight force,

leaving the distortion to be resisted by the other brace. As the other diagonal tends to lengthen, the brace in that line, if a tie, will come into action; but, if it is a strut with abutting ends, it cannot exert tension, and the first diagonal must supply the force. If, however, one diagonal is a tie and the other a strut in the same panel, or if both diagonals are competent to carry either kind of stress, they will both act together; and the distribution of the given shear between them is indeterminate, except on some assumption or condition. No good design will be so constructed.

24. **Stresses in the Verticals.** — Since *e s* is the vertical component of the stress in S F, it must be the compression in S E; for the tension in S F, upon reaching S, is decomposed, the horizontal component compressing S R, and the vertical component compressing S E. The vertical force of S E will next pass over E B, with the addition of that part of the weight at E which may properly pass that way. As these things are true of any set of diagonals and verticals, it follows that *the ordinate to the shear curve at the middle of a panel determines the stress in that diagonal and vertical*, *which*, taken together, may be considered to *connect two loaded joints.* If, therefore, we imagine the load to be upon the top chord in place of the bottom chord, *s e* will give the compression on R F instead of S E; for, considering the vertical force passing from R to S over the pieces R F and F S, there is then no applied load between R and S to alter its amount. In Fig. 6 the stress in B A equals the ordinate at *t*: were the load on the top chord, the stress would be increased by the half load at B, so that it would become *b a.* A change of load from one chord to the other affects the stresses in no pieces except the verticals, and only changes the *amount* of those stresses, but not the *kind.*

25. **The Middle Vertical.** — The middle vertical alone may sometimes offer an exception to the preceding way of determining the maximum stress in it. It will be seen, that, when the load is upon the upper chord of Fig. 6, and the entire span is covered with rolling load, the ties which meet at I in the lower

chord will be in action, since the bridge is symmetrically loaded; and that the vertical at I must then transmit from the top to the bottom joint the entire weight which rests at its top, or one panel weight of steady and rolling load. If this amount is more than the ordinate which the shear diagram gives as belonging to this vertical, the former must be taken in place of the latter. The same remark holds if the load is on the bottom chord, and the main braces lie in the other diagonals; in which case they will be struts. The verticals will then be tension members. If the truss is as here represented, or if, when strut diagonals are used, the load is on the top chord, no special attention need be paid to the middle vertical, as it then follows the general rule.

26. **Stresses in the Chords.**—As the equilibrium polygon A′ F′ I′ K′ D′ alone is needed for determining the maximum bending moments at all points, we have simply to multiply each ordinate under a joint by the value of H from the stress diagram, and divide by the height of the truss, to obtain the chord-stresses. For suppose that we pass a vertical plane of section through the joint F. It follows from § 5 that the moments of the stresses in all of the pieces cut by that section, when taken about any point in the section, must balance, or, in other words, equal, the bending moment at that section. If the origin of moments is at F in the lower chord, and the moment of resistance of all the stresses on the left of the section is desired, as balancing the bending moment of the external forces on the right, we see that the vertical R F, the diagonal S F, and the chord piece E F, terminate at the proposed moment axis or point F; and hence, having no arms, their stresses have no moments. There only remains the compression in the piece S R of the top chord, multiplied by R F, the perpendicular to it from F. As this moment must resist the bending moment, we divide the latter by R F to find the stress in S R. Hence follows the opening statement of this section. Similarly, taking moments round R, and conceiving them, as taken on the right of the section, to balance the equal bending moment on the left, all the pieces which meet at R have no moments; and there

remains the tension in F G multiplied by R F, its lever arm. The same steps will give the chord-stresses in the other panels. As, with a uniform load from A to D, no counters are required, the shear diagram being simply *a b v c d*, or the tendency of the panels to change their shape being such that the ties which slope from G and J toward the centre will slacken, the counters may be disregarded in determining chord-stresses.

27. **Chord-Stresses** (*continued*). — It follows that the compression in S R equals the tension in F G; and the same equality is true of B S and E F and of the other portions of the chords. This fact may also be readily seen if we pass an oblique plane of section through S R and F G: the only other piece cut by the section will be R F; and as this vertical cannot carry a horizontal stress, and the external forces are all vertical, the stresses in S R and F G must be equal and of opposite kinds, so that those pieces of top and bottom chord which lie between two diagonals in action will have the same amount of stress. That the upper chord is in compression, and the lower chord in tension, follows from the tendency of the load to make the truss concave on the upper side, or to shorten the top chord and lengthen the bottom one. Of course the compression or tension is constant for a panel-length, and the stresses are symmetrically distributed with regard to the middle of the span. As there is no bending moment at the abutments, there is no stress in A E or L D; and these pieces are unnecessary for the equilibrium of the truss. If the bridge-seats came directly under the top chord, A E and L D might be left out; but, as it is, they lend aid to the bottom lateral bracing in stiffening the bridge against wind and vibration. The counters being neglected, it is evident that two panels of the top chord at the middle have the maximum stress, and that no equal tension occurs in the bottom chord. The reverse is true if the diagonals are struts.

To find, therefore, the tension and compression in the bottom and top chords between any two adjacent diagonals which incline the same way, it is only necessary to select the ordinate to

the maximum equilibrium polygon under the common panel joint of the two chord pieces, multiply by H, and divide by the height of truss. Since H is constant, and the height of the truss, in trusses with parallel chords, is also constant, if H is numerically some simple factor or multiple of the height of the truss, the whole operation can be performed by changing the scale by which the ordinate is measured.

28. **Parabola for Chord-Stresses.**— Availing ourselves of the last suggestion, we are able to draw a figure for chord-stresses at once. That the vertices of the equilibrium polygon for an equal load on each joint of a bridge-truss lie on the parabola which is the limit of the polygon when its sides become infinite in number, or, in other words, when the load is uniformly distributed over the truss, may be proved by taking moments at any joint. If, then, we draw a parabola below the truss whose middle ordinate is equal to the maximum stress in the chord at the centre of the span when the load is uniformly distributed over the span, the ordinates at the several joints will give all the chord-stresses directly.

As the bending moment at the middle of a beam which is uniformly loaded with a total load W'' is equal to the supporting force at one abutment, $\frac{1}{2}W''$, multiplied by its arm, one-half the span $= \frac{1}{2} l$, minus the weight on the half span, $\frac{1}{2}W''$, multiplied by its arm reckoned from its centre of gravity, $\frac{1}{4} l$, or

$$M = \frac{1}{2} W'' \cdot \frac{1}{2} l - \frac{1}{2} W'' \cdot \frac{1}{4} l = \frac{1}{8} W'' l,$$

or is equal to one-eighth of the total load multiplied by the span, the maximum stress in one chord of a truss of the height k will be, at the mid-span,

$$\frac{W'' l}{8 k},$$

and this will be the value of the middle ordinate.

Therefore proceed as follows: On a horizontal line A B, Fig. 8, equal to the length of the span, lay off the panel joints D, E, F, G, &c. Draw C I and A P vertically, each equal to $\frac{W'' l}{8 k}$, C I being at the middle of the span. Divide A P into

the same number of equal parts with A C; thus finding the points K, L, N, and O. Draw I O, I N, I L, I K. The points R, S, T, and U, where these lines cut the verticals dropped from the panel joints, will determine the desired ordinates C I, G R, F S, &c., which are the stresses in the chords from the middle to the abutment. Since the lines which radiate from I cut from the vertical P A the distances P O, 2 P O, 3 P O, &c., and the points R, S, T, &c., occur at horizontal distances of C G, 2 C G, 3 C G, &c., from I, it is easy to see that the vertical distances of R, S, T, &c., from I, vary as the square of their horizontal distances, and that these points will, therefore, lie in a parabola. The other half of the figure may be completed if desired.

Figs. 7 and 8 will thus give at once all of the desired vertical and horizontal forces in parallel-chord trusses with one system of bracing. The diagonal stresses will be obtained by drawing lines as described in Fig. 7, parallel to the braces. For the small number of necessary lines, and the exactness of intersections and measurements, it is believed that these two diagrams will compare very favorably with any others. In Fig. 11 they are given with no repetitions or useless lines, to show how small is the required work.

29. **Sections of Pieces.** — Each part of the truss which undergoes tensile stress should have its effective or smallest section equal to the quotient of the maximum stress it must exert divided by the safe working stress on the square inch. The cross-section of pieces in compression should be determined by Gordon's Formula, or some similar method. The working out of details does not come within the scope of these pages, and we will refer the reader to books which treat of the strength and resistance of materials. Some additional matter which appropriately comes in this connection may be found in Part I., "Roofs," Chap. VIII.[1] Pieces liable to alternate compression and extension should be made of larger cross-section than for

[1] Any reader who is taking up this subject for the first time, and unassisted, will find much valuable information, presented in a simple form, in Boller's Iron Highway Bridges, published by Wiley and Sons, New York.

one stress only, as the safe working stress on the square inch is less. This safe stress is also taken as a smaller quantity for web members near the middle of the span (or for any others which experience the maximum stress, especially if accompanied by shock, every time a load passes across the bridge) than for those members which only have the greatest stress when most or all of the bridge is loaded with the heaviest possible load; a contingency which occurs less often in large spans than in those of moderate length.

30. **Moment Diagram for Partial Moving Load.** — If a load W is placed at any point C, Fig. 9, distant x from one support of a beam A B whose span is l, the equilibrium polygon will be A′ C′ B′, and the stress diagram will be drawn on the line 1–2. The supporting force at A, obtained from similarity of triangles in the stress diagram and the equilibrium polygon, is $W \frac{l - x}{l}$, and the bending moment at C will be $\frac{W}{l} (l - x)\, x$. As this moment varies as the product of x and $l - x$, the two segments into which C divides the span, and as the maximum ordinate occurs at the weight, the maximum moment at each point of the beam, as the weight W rolls across it, will be represented by the ordinate to a parabola, drawn through A′ and B′, and having its vertex at a distance $\frac{W\, l}{4\, H}$ or $\frac{1}{4}\, W\, l$ vertically below the middle of the beam, according as the ordinate is or is not to be multiplied by H to give the bending moment. This parabola is shown by the dotted curve A′ D′ B′.

If the load is distributed over a certain space C F, in place of being concentrated at a single point, we can place the load first on the middle of the span, and, by a stress diagram or otherwise, draw A′ D′ B′. The diagram is 2′ 1′ 0′. The lines 2′–0′ and 1′–0′ will give B′ D′ and A′ D′, which, as explained in § 10, will meet on the vertical dropped from the centre of gravity of the imposed weights. If the centre of gravity of these weights is placed at the middle of the span, and they then occupy the positions C, E, D, and F, the true equilibrium poly-

gon will be A′ E′ G′ F′ B′, of which the part E′ F′ is readily constructed from the points of division on 2′–1′. The maximum ordinate is at G′, and is less than that for the concentrated weight by the quantity G′ D′. When the loads are moved until their centre of gravity is over I, the polygon passes through K, and it is seen that the maximum ordinate will always be under the centre of gravity of the load as it passes across the span. The locus of the extremity of this ordinate will lie within the previously-described parabola at a *constant vertical distance* above it; and this curve, shown on the right, is therefore readily drawn. For, by reference to § 10, it is seen that the amount the polygon runs within the prolongation of the tangent at any point depends upon the bending moment of the loads passed by; and, as the subtractive moment will always be a constant quantity, the vertical distance G D′ or K I is constant. When the load begins to pass off the span at either end, the two curves approach one another.

Such a moment diagram as Fig. 9 may be useful when the requirements for a highway bridge are, that it shall support a certain distributed load, or a certain other load concentrated on the wheels of a wagon. The diagram also applies to a locomotive in connection with a lighter train: it will be referred to again.

31. **Shear Diagram for Partial Moving Load.** — The diagram for shear for a single load at the middle of the span will be the two equal rectangles *a i h d* and *d g k b* of Fig. 10, the height *a i* and *k b* of each being one-half the load. If the load is moved to *c*, we have *t s u v*. If it is put at one-quarter of the span from one end, one rectangle will be one-quarter of the load in height, and the other three-quarters of the load: hence we see, that, as the load passes across the span, the maximum shear at successive points will be found by drawing ordinates to an inclined line which reaches from one extremity *b* of the span to a point *l*, at a distance *a l* equal to the load, above the other extremity *a*. If the load is distributed according to any law over a definite area, the shear line will fall, according to the

intensity of the load, over that portion of the span which is loaded. Thus, for the four concentrated loads at *c*, *e*, *d*, and *f*, corresponding to those of Fig. 9, the shear diagram changes to *i r q p o n m k*. If this load had been spread over the portions from *c* to *f*, the dotted line would have taken the place of the broken line; and, as before, it can be used for the broken line, if the ordinates are measured at points midway between the weights. The maximum ordinate is now just before *c*, but still equals *a i* or *d h*, the ordinate at the centre of gravity. Hence, when the system of loads is moved, as there is always a definite amount of load in advance of the centre of gravity, the maximum ordinate will be found by drawing a line parallel to *b l*, and at a distance below it determined by the position of the end of the load, as in this figure at *r s*.

32. **Stresses due to Locomotive.** — While it has thus far been considered that the load on a truss, or the weight of a train, is uniform per foot, it is manifest that the locomotives at the head of a train are heavier per foot than the cars, and that such additional weight should be allowed for, more especially in designing the bracing. If we find the *excess* of such load over the previously-considered uniformly-distributed load, and then draw moment and shear diagrams for this excess of load, distributed over its proper space as it moves across the span, these diagrams can be added to the ones previously obtained, and the solution will thus be completed. As the maximum moment for a concentrated load occurs when the weight is placed at the middle of the span, there is not so much need of adding its diagram to that for uniform load. At the time the load in question is in the middle of the span, the train can cover but half of the bridge; and, by the time the train covers the greater part or the whole of the span, the locomotive is just leaving the bridge, and the ordinate for the weight in that position will be very small. If, however, some car of the train is liable to carry a heavy load, — such as a locomotive *in transitu*, heavy castings, or ordnance, — this diagram is very properly added. The shear diagram for concentrated weight is particularly applicable to railroad bridges.

These two diagrams are added to the usual ones in Fig. 11. In adding them to figures already drawn, we must see that the moment diagram has the same value of H as the large one to which it is joined; or, if the ordinates themselves represent chord-stresses, that the scales shall be the same, and that the two shear diagrams shall have the same vertical scale.

CHAPTER III.

TRUSSES WITH HORIZONTAL CHORDS. — SINGLE SYSTEMS. — VARIOUS TYPES.

33. **General Example.** — In the illustrations annexed to Fig. 11, the span is taken as 120 feet, height of truss 15 feet, rolling load for one truss 1,000 lbs. per foot, steady load for same 500 lbs. per foot. The maximum ordinate for chord-stresses will be, by § 28,

$$\frac{(60{,}000 + 120{,}000)\ 120}{8 \times 15} = 180{,}000 \text{ lbs.}$$

N O is made equal to this amount at the middle of the span, and P Q equal to the same at the abutment. P Q is divided into the same number of equal parts (here four) as there are panels in the half-span; and lines from O to these points of division cut off the chord-stresses at N O and the points S, U, and W.

The abutment reaction of truss and complete load will be half $(60{,}000 + 120{,}000) = 90{,}000$ lbs.; and of truss alone, half $60{,}000 = 30{,}000$ lbs. Lay off these quantities at $s\,t$ and $p\,q$ respectively, connect the points t and q with o, the middle of $s\,p$, drop verticals from panel joints to these inclined lines, bisect the parts so obtained, connect the points of division on $t\,o$ and $o\,q$ as directed for Fig. 7, and the shear curve is completed for the above load.

If the train may contain or be drawn by an engine having 56,000 lbs. on three pairs of drivers, 14 feet wheel base, the load for one truss will be 28,000 lbs. on 14 feet; from which we must deduct 14,000 lbs. mean rolling load already consid-

ered, and we have 14,000 lbs. extra weight to be allowed for. We may consider this load as concentrated on one panel joint, or as distributed to the adjacent joints. The first supposition will give the most stress. The chord-stress at the middle will be, by § 30,

$$\frac{14{,}000 \times 120}{4 \times 15} = 28{,}000 \text{ lbs.},$$

which is plotted above the horizontal line, at N Y, and the parabola completed. The combined chord-stresses are thus obtained by scaling from the vertices of one polygon to the corresponding vertices of the other, Y O, R S, T U, and V W.

For shear, lay off the excess, 14,000 lbs., at $s\,r$, below the line $s\,p$, and draw a straight line from r to p. As the ordinates to the shear curve give stresses in the web when the load extends from the right abutment to the panel in question, it is apparent that this construction will give at the panel joint the additional amount of shear due to the engine at the head of the load. As the shear is measured in the middle of each panel, move $r\,p$ parallel to itself until the point p on $r\,p$ falls in the middle of the first panel on the right: the alteration is too small to be seen in this figure. By laying the diagram off below the horizontal or base line, while it is still considered positive, the two ordinates at any panel are at once combined. Lines drawn parallel to the braces, from the upper ends of these ordinates, and limited by horizontal lines from the lower ends of the ordinates, as shown in the figure, will give the stresses in the braces. It will be seen that the distance over which braces that incline one way extend is prolonged farther beyond the middle; the limit now being fixed by the point u, where the shear curve crosses the line $r\,p$.

Remember, that, if the *additional* load is not concentrated at one joint, the new lines for chord-stresses and shear will be a certain constant, vertical distance within the ones just constructed, as explained in §§ 30 and 31.

34. **Howe Truss.** — Truss I of Fig. 11 is generally called,

from the inventor who built it in wood, the Howe truss. The diagonals are struts, and the verticals ties (see § 4). The load is upon the bottom chord. If the verticals are iron rods, and the remainder of the bridge is built of wood, this type of truss is well adapted for such materials. Iron bars used in the bottom chord bring the truss into the class usually known as combination bridges. The end posts and end pieces of the top chord have no truss-stresses, but are useful in connecting the top lateral bracing by a stiff frame with the abutment. As generally constructed, the joint at B has not sufficient rigidity for this purpose; so that a pier or abutment panel is introduced. Some designers, however, make a special connection at B, and carry the top chord no farther, the truss then ending as shown at the right.

The stresses in the bottom chord, in the successive panels from the abutment to the middle, will be equal to V W, T U, R S, and Y O. In the top chord the compressions will be, from B to the middle, V W, T U, and R S. The compressions in the diagonals which have corresponding letters are *a b*, *c d*, *e f*, *g i*, and *k l*. For convenience in screwing up the bridge, and to stiffen the main braces by bolting to the counters where they intersect, most builders carry the counters through all of the panels: they may be of small cross-section. As these bridges are generally covered from the weather, and have, in some rare instances, been blown from the abutments by violent gales, they should be bolted to the abutments, when the counters will resist any sudden gust of wind from beneath. The tensions in the verticals B C, D E, F G, and I K, will be the stresses represented by the vertical lines at *b*, *d*, *f*, and *i*. The middle vertical, when the truss is completely loaded, and G I and I M are under stress, must carry one panel weight $= 15 \times 1{,}500 = 22{,}500$ lbs. This amount is less than the ordinate at *i*, and therefore of no consequence. By symmetry we get the stresses on the other half of the truss.

35. **Pratt or Quadrangular Truss.** — Truss II differs from the truss of Fig. 6 in being loaded on the top chord. The

modification at the right end of this truss may be introduced with advantage, in case it is not too expensive to carry the masonry for the bridge seats to such a height. If the bridge seat is at the level of the lower chord, the end post will carry one-half the *total* weight of truss and rolling load, or $r\,t$; for the half-weight at A will come upon this post in addition to the shear in the first panel.

Truss III differs from the truss of Fig. 6 at the ends only. The end diagonal is a strut, which transmits the same amount of force as do the diagonals in the first panels of the preceding trusses. This introduction of *inclined end posts* is very common in iron bridges, and is economical. The first vertical now becomes a tie, which manifestly can carry only a panel weight of steady and rolling load. The stress in the bottom chord will be uniform from A to D, and equal to V W, the stress which would have been found in B D of the truss above.

Authors and engineers are not agreed upon the name by which the type of truss illustrated by Fig. 6 and Trusses II and III shall be designated: it is often named after some designer who invented a modification of one or another detail of construction, with which details a skeleton diagram has nothing to do. Some of the names which have been given are Pratt, Whipple (see § 48), Murphy-Whipple, Linville, &c. We prefer the first title, if it is to be named after any engineer, but would rather style it a *single quadrangular truss*, by which shall be understood a truss with horizontal chords, inclined ties which extend but one panel each, and vertical struts. The modification of Truss III is understood when the qualifying term *inclined end posts* is added. If the rolling load comes upon the lower chord joints, the bridge is called an *over-grade* or *through* bridge; if upon the upper chord, an *under-grade* or *deck* bridge.

36. **Comparison of Trusses.** — Truss IV, a Howe truss, is seen to differ from Truss I in having its load upon the upper chord; otherwise no separate remarks are necessary. The Howe truss is evidently the reverse of the Pratt or quadrangular truss.

As stated before, it will be seen from these examples that no change in the magnitude of the stresses in *chords* or *diagonals* occurs when the load is shifted from the bottom to the top chord; and hence, so far as they are concerned, it makes no difference whether the steady or rolling load is considered as applied to either or both at once. This truth may also be recognized from the fact that neither the bending moment nor the shear can be changed by moving a load vertically. The stresses on the *verticals*, however, will be altered by such a change, as may be seen by remembering that the vertical and diagonal, which together connect two adjacent weights, transmit the same amount of vertical force.

37. **Warren or Triangular Truss; every Joint Loaded.**— The truss marked V is usually known as the Warren Girder; although the name Triangular Truss is often used, and is appropriate. Every joint of the bracing is loaded; the loads on the bottom chord at B, E, &c., being transferred to C, F, &c., by the vertical suspending rods B C, E F, &c. These rods carry whatever load can be placed at their lower ends. No part of one chord has the same stress as any part of the other chord; but a section at successive joints, by the usual analysis, necessarily determines the stress in the opposite pieces of the chord: thus the ordinate of the diagram for chord-stresses, which comes above C, will be the tension in A D; the ordinate above D will be the compression in C F, &c. The stresses in the braces will be found as usual; but, by reason of the alternating inclinations of the pieces, they will be successively compression and tension from the abutment to the middle. The same rule will hold beyond the middle as far as counters may be necessary; so that a certain number of the inclined members must be struts and ties alternately, as they change from main to counter-braces under a passing load. In this particular example one piece each side of the middle must be designed to resist tension as a main brace, and a small amount of compression as a counterbrace. As it is of doubtful economy to have one piece repeatedly undergo a reversal of stress, some builders use a hollow

strut with light tension rods within it when a main-compression member has to act as a counter-tension member. Tension bars are stiffened against a moderate compression by a light lattice bracing of flat bars riveted to the edges.

Sometimes, though rarely, this truss is loaded upon the top chord, when this figure inverted will represent the case: B C, E F, &c., must then be replaced by vertical struts, and the series of inclined members will begin with a tie.

38. **The Same; alternate Joints Loaded.** — When the verticals are omitted, the load will come upon alternate joints, as seen in Truss VI. The shear ordinate will be measured midway between the loaded points, as usual, and belongs to the diagonals on each side of it; so that the diagonals in pairs will have the same amount of stress: that is, the compression in A B equals the tension in B C, the compression in C D equals the tension in D E, &c. For this truss also, K L will be a strut, and L M a tie, when the rolling load extends from the right abutment to M: but, when the load covers A K only, these pieces undergo stresses equal to those in G I and I K for a load on the right segment up to K; K L will then be a tie, and L M a strut. Web members near the middle of the truss must therefore be adapted to both compression and tension. Lines from *b*, *d*, *f*, *i*, and *l*, in the shear diagram, parallel to these braces, will give the amount of stress in each, as usual.

The stresses in the pieces of the top chord will be given by the ordinates over the corresponding joints of the bottom chord, as was done for both chords of Truss V. But, to find the tension in A C of the bottom chord, the plane of section must pass through B; and the ordinate will therefore be the one at B′ in the diagram below the truss. The tension in C E will be given by the ordinate at D′: hence the ordinates at B′, C′, D′, E′, &c., will be the stresses in those pieces of the chords which lie above them. The funicular polygon is drawn as usual, with angles under the loaded joints; but additional ordinates are drawn to the middle points of the sides.

When the top chord is to be loaded, the truss may stand as

here sketched, or be inverted, and the web system will then begin with a tie. Shifting the loads from the bottom to the top chord moves them laterally half a panel, if the truss is not inverted; and hence the chord-stresses are changed: the method of analysis is not affected, however.

39. **Comparison of Trusses.** — We show in the annexed table, in one view, those parts of the six trusses which have the same stress. The first portion of the table is devoted to the chord pieces which correspond to the ordinates V W, T U, R S, and Y O; and below them will be found those web members of each truss whose stresses have *vertical components* equal to $b\ v$, $d\ w$, $f\ x$, $i\ y$, and $l\ z$. The span and the load are in each case the same.

Truss Ordinate.	I.	II.	III.	IV.	V.	VI.
V W =	A C B D	A C B D	A D	A C B D	A D	B D
T U =	C E D F	C E D F	C E D F	C E D F	C F	D F
R S =	E G F I	E G F I	E G F I	E G F I	D G	F I
Y O =	G K	G K	G K	G K	F K	I L
$b\ v$ =	A B B C	A B B C	A C	A B	A C	A B B C
$d\ w$ =	C D D E	C D D E	C D	B C C D	C D	C D D E
$f\ x$ =	E F F G	E F F G	D E E F	D E E F	D F	E F F G
$i\ y$ =	G I I K	G I I K	F G G I	F G G I	F G	G I I K
$l\ z$ =	K L	K L	I K K L	I K K L	F G G K	K L L M

We have added to the trusses just discussed two other types, resulting from efforts of the designers to invent trusses which shall carry additional loads without an increase in the number of panels; that is, when the panel joints become so far removed from one another, by reason of the height of the truss combined with the desired angle of inclination of the diagonals, that it is thought best to concentrate loads at intermediate points, they

have introduced short auxiliary pieces to support such loads more or less directly.

40. **Baltimore Bridge Company's Patent Truss.** — Taking up first the truss marked VII, and drawing the diagrams marked Fig. 12, we see that A C, C E, E G, and G K, of the top chord, will have stresses equal to the ordinates B′ F′, C′ G′, D′ I′, and E′ K′, which are here drawn on the other half of the span to accommodate the shear diagram. It is also evident that the vertical force in M B, O D, Q F, S I, and U W, will be given by *b n*, *d p*, *f r*, *i t*, and *l v*, all of them being tension members. M N, O P, Q R, &c., will have a tension of whatever weight is placed at N, P, R, &c. When the moving load extends from the right abutment to N inclusive, the shear in the panel L N will be *a m;* and this shear will be distributed between A M and L M, extending the former, and compressing the latter. At that time the shear in M B will be less than *b n*, since a load is at N; and its amount will be determined by drawing a line through *a* parallel to *x u;* which line is, in this case, *x u* itself. By referring to the enlarged sketch, Fig. 13, it will be seen that the distance from *n* to the point *r*, where this line cuts *n b*, will be the shear then existing in M B. If *r s* is drawn horizontally, making $s\,m = r\,n$, *s q* drawn parallel to M B will be the stress in that member at present, while *a s* is the load at N which passes over L M and A M together: hence *q t* will be the tension in M A, and *t a* the compression in L M. Or, by other reasoning at the point M meet four pieces: the tension in M N is known that in M B has just been determined: the other stresses will be found by completing the quadrilateral in the order *a s*, *s q*, *q t*, *t a*. The horizontal component of the stress in L M will be the tension in L B, while the vertical component of the tension in A M will be the compression in A L.

In the same manner, for the panel B D, the ordinate *c o* and the portion of *d p* cut off by a line through *c*, parallel to *x u*, are to be used. Fig. 13 will illustrate this case also. The stress in C B will then be obtained from that in C O. As it is apparent, however, by a sketch like Fig. 13, that we shall always have *a s*

equal to a panel weight, since $m\,n$ is a panel distance, therefore L M, B O, D Q, &c., each carry one-half a panel weight as a vertical component; and A M, C O, E Q, &c., will carry one-half panel weight less than the shear $a\,m$, $c\,o$, &c., in their respective panels: hence, upon passing the centre I, K U will be necessary as a counter, if the ordinate at k is more than one-half panel weight. As B O and D Q are the only pieces which from their inclination prevent the tensions in B D and D F from being equal to the compressions in A C and C E respectively, the stress in a bottom chord piece will be found by adding the horizontal component of $t\,a$, Fig. 13, to the stress in the proper upper chord piece.

If the polygon for chord-stresses had been constructed with angles under every loaded joint, we might also determine the bottom chord-stresses from that; for, if C represents the compression in a certain panel of the top chord, and T represents the tension in the same panel of the bottom chord, while k denotes the height of the truss, we shall have, if we make a vertical section through, and take moments about, any such joint as O, the moment of resistance $= \mathrm{T} \cdot \frac{1}{2}\,k + \mathrm{C} \cdot \frac{1}{2}\,k$. If the ordinate under this joint represents chord-stress in place of bending moment, it will already have been divided by k: hence the ordinate below O denotes half $(\mathrm{T} + \mathrm{C})$ for the pieces B D and C E. But the compression C in C E is given by the ordinate below the joint D; hence twice the ordinate below O minus the ordinate below D will be the tension in B D: or, from the ordinate below O subtract the amount by which it falls short of the ordinate below D, and the stress in B D will be obtained. The process may be repeated for the other pieces of the bottom chord.

The inclined portal at the right end of this truss is a modification which affects only the end panel, as before noted. It will be useful to recall the points developed in Part I., "Roofs," that any alteration of pieces in a panel cannot affect parts outside of that panel which are not displaced by the change. Since Truss VIII is Truss VII inverted, bringing the load on

the upper chord, the analysis will be similar, and need not be repeated.

41. **Kellogg's Patent Truss.** — In this truss, marked IX, the tension in M N will be given by the ordinate for that joint, or B′ F′; the compression in C E and the tension in D O will equal C′ G′; and in the same way D′ I′ applies to E G and F P. The middle ordinate belongs to the middle panels of the top chord. The ordinate *a m* will be the shear in A C; *d p*, the shear in C D; *f r*, the shear in E F; *i t*, that in G I; but *k u* gives the shear in the counter K L. The vertical component of the stress in M C, C N, E O, and G P, is the amount of weight at M, N, &c. The horizontal component will be resisted by the lower member. The tension in A M will be the horizontal component of the compression in A C; the stress in M N has already been indicated; that in N D will be greater than the tension in M N by the horizontal component of the stress in N C. The stress in O F is increased by the same amount over that in D O. The force in B C equals the weight at its foot. The compressions in D E, F G, and I K, are given by *e q*, *g s*, and *k u*. A right-angled triangle whose hypothenuse is parallel to C N, and whose altitude equals a panel weight at N, may be drawn to give the stress in C N and the horizontal component to be added for the stress in N D, or the three forces may be compared by the similar triangle B C N.

Any other parallel-chord trusses with single systems of bracing can be analyzed in a similar way.

42. **Plate - Girders.** — Where, from lack of room, we are obliged to make the ratio of the height of a truss to the span small, and where the load per foot to be carried is large, as is the case with short span railway bridges required to carry a locomotive whose drivers will cover a considerable portion of the span, a plate girder may be the most desirable form of bridge. In this case it may be conceived that the web members have been made so thin in the dimension tranverse to the span, that they occupy with their other dimension the whole of the web. It is still supposed that the plate of which the web is

now formed withstands all of the shear, and that the moment of the stresses in the two flanges resists the bending moment. As the web is quite thin, this assumption involves no material error. The web is usually of one thickness throughout the span, and hence should be thick enough to withstand the shear at the abutments: it should also be thick enough to resist compression in a direction of 45° with the horizon, of an amount at each section equal to the shear at the same section. If the web, computed by Gordon's Formula, as a column of 1.4 times the height of the girder, and of a thickness required to resist the shear, is not thick enough to carry this thrust, vertical stiffening ribs of angle or T iron are usually riveted to the web at such a distance apart as shall reduce the length of the compressed piece, measured at an angle of 45° between two verticals, to the proper amount for the given thickness and compression. It results that usually no verticals will be required near the middle of the span, but that there will be a less and less interval between these stiffening ribs as we approach the abutments. From practical considerations, the web is generally not less than ¼ inch to ⅜ inch thick, — an amount usually enough to resist the shear.

In determining the cross-section of the flanges, as the girder is frequently of a uniform height from end to end, and the flanges of a constant width, we see that the thickness of the plates of the flange at each section will be directly proportioned to the bending moment or chord-stress. They may, therefore, be laid off on the parabola for bending moment, and their lengths determined. Thus, suppose that the tension or compression in one flange at mid-span, from a uniform load, is 100,000 lbs., that the flange is 8 inches wide, and that the safe stress on the square inch is 8,000 lbs. Every 16,000 lbs. height of ordinate will require ¼ inch thickness of flange. If, therefore, we draw the parabola of Fig. 14 to represent the chord-stresses, and draw horizontal lines at intervals, in height, of 16,000 lbs., the length of the successive plates of the flanges may be easily measured off on the scale of the base-line which represents the

span. The thickness of the plates may be made ½ inch, ⅜ inch, and ¼ inch, or as judgment dictates. The lengths of the pieces can now be easily arranged to break joint, and the required cover plates added above to supply the deficiency at joints. When cover plates come near together, they may be combined in one supplementary plate.

Fig. 15 is intended to show how the several pieces of the flanges and web should break joint. The rectangle at A represents the web 3′ 6″ deep, and ⅜″ thick, vertical angle irons being placed on the joints and the dotted lines. B gives the bottom angle irons, the top set being similar, and all $3\frac{1}{2}'' \times 3\frac{1}{2}'' \times \frac{1}{2}''$. At C is shown the bottom flange, the top flange being the same, 12″ wide ⅜″ thick for 40 feet, and ¼″ thick for the strip which extends across the whole span. The cover plates for flanges are 26″ long. The above dimensions are from an existing girder of 65⅝ ft. span.

43. **Truss of Odd Number of Panels.** — Trusses which have a single system of bracing are usually designed with an even number of panels. But it is evident that an odd number of panels may be used if desired: indeed, as we shall see presently, trusses with a double system of bracing must have an odd number of divisions in one system. The truss of Fig. 16 has five panels and four loaded points, not including the points of support. The equilibrium polygon or the diagram for chord-stresses is drawn below the truss; and it is seen that the maximum ordinate C′ O′, as obtained by § 28, will not represent a chord-stress. But if we divide B′ P into five equal parts, corresponding to the number of panels in the *whole* truss, or twice the customary number of parts, and connect alternate points of division with O′ by O′ Q and O′ R, the ordinates I′ K′ and L′ N′ will be the chord-stresses; I′ K′ existing in E N and F I, and L′ N′ being found in I L and N O. The shears will be found as usual. The diagonals G I and F K cross the middle of the span, are not under stress for a complete load, and are properly counter-braces. The panel G F I K is sometimes known as a *cross-panel*.

44. **Trapezoidal Truss.** — Before closing this branch of the subject, we will allude to two or three trusses which should not be overlooked. The trapezoidal truss may be considered a truss of three panels, and will be made from Fig. 16 by removing two panels. Its analysis is, therefore, the same. It is also easily treated by a stress diagram, as applied in Part I., "Roofs;" and the reader is referred to Fig. 13 and p. 21 of that part. As the only piece strained to a maximum by a moving load is the counter, or brace in the middle panel, two diagrams — one for a complete load, and one for moving load covering one joint — will suffice.

45. **Fink Truss.** — This type of truss, illustrated by Fig. 17, and invented by Albert Fink, does not properly belong to the class of this chapter, as it has no bottom chord. When the travelled way passes at a level with the feet of the main posts, there may apparently be a bottom chord by reason of the arrangement of the lower lateral bracing; but such a member is not concerned with the supporting power of the truss. An inspection of the figure will show that each short post will carry one panel weight; each post at the quarter-span, two panel weights; and the middle post, four panel weights. The stress in each tie must have as a vertical component one-half of the weight on the post which it supports. The compression in the top chord will be uniform from end to end. A moving load covering the entire span will cause the maximum stress in every piece. If a section is made at the middle of the span, the stress in the main or longest tie may be obtained by taking moments about the middle joint of the upper member; and, that stress being known, a section at the quarter-span will give the stress in the next tie. But moments taken about the bottom of the centre post will not give the compression in the upper member, as the stress from the ties which meet at the middle joint of the upper member must be added.

A modification of this truss has been used, which is shown in Fig. 18; and there is a bottom chord in this design. We have drawn a half-span, and will leave its analysis, which presents no special difficulty, as an exercise for the reader.

46. **Bollman Truss.** — The truss of Fig. 19, designed by Wendell Bollman, also requires no bottom chord. If the load is upon the lower line, the portion of load at each apex is distributed to each abutment directly by the two independent ties which run from each loaded point. The bridge being loaded throughout, the compression in the upper member will be the aggregate from the several triangular systems, and will be uniform throughout. The verticals will theoretically have no duty to perform: practically they keep the upper member from bending under its own weight and the thrust. If the load is on top, the posts will each carry a panel weight. The inventor added diagonals in each panel, giving a separate means of support for the foot of each post. Most of the ties run at too small an angle with the horizon, and their section will be large for the amount of vertical component which they convey to the abutments. The truss is not an economical type, therefore, and has been but seldom built.

47. **Wind Bracing.** — The wind exerts a thrust against the side of a truss, generally in a horizontal direction; and this thrust must be allowed for. Wooden bridges which are covered from the weather will expose a large surface to the wind; but, in any case, the total pressure will be obtained by multiplying the area of exposed vertical surface by the greatest intensity of the wind pressure per square foot, the latter being taken at from forty to fifty pounds. To the lateral surface of the truss add, for an open or a deck bridge, the area offered to the wind by a train of cars. The wind pressure on the bridge will be a uniformly distributed force, acting horizontally; and it will be resisted by a system of trussing between the two upper chords and also the two lower chords of the bridge, arranged upon any convenient plan. Usually the Howe Truss type of struts diagonal and ties perpendicular, or the Pratt type of ties diagonal and struts perpendicular, to the chords, is used. Both diagonals will be required in each panel of the lateral bracing, as the wind may blow from either side. The pieces of the bracing should, properly, increase in section as they approach

the abutments. Quite often they are made of one uniform section throughout the bridge.

As the cross-section of a bridge is a hollow rectangle, the thrust of the wind tends to rack it more or less. Vertical lateral bracing can be introduced in a deck bridge, should never be omitted when the truss is supported at the ends of the lower chord, and should meet the chords at the same points with the horizontal lateral bracing. In other cases, especially in through bridges, short knee-braces or gusset-pieces are used to give stiffness to the rectangle. Sometimes the verticals, if struts, are spread at the bottom transversely to the truss to aid in giving stability. The wind pressure has an overturning moment equal to the total pressure multiplied by the vertical distance of the point of application of the resultant wind pressure above the abutment; and this is resisted by the weight of the bridge multiplied by its half-width between centres of trusses, or by the tension of bolts which may fasten the truss to the masonry.

Maximum Stresses from Concentrated Moving Loads. See Appendix, p. 178.

CHAPTER IV.

TRUSSES WITH HORIZONTAL CHORDS. — MULTIPLE SYSTEMS.

48. **Double Quadrangular Truss.** — When, from length of span and corresponding economical height of truss, with a single system of bracing at the usual inclination, panel joints come too far apart, it is frequently the practice to add another independent system of bracing. Such trusses will now be discussed. The truss of Fig. 20 is the first one to be taken up. Where the compression members of the web are vertical, and the ties inclined, there seems to be a general agreement that an inclination, for the latter members, of 45°, is the most economical: hence follows this type of truss when the height is much more than a desired panel length. The name of *double quadrangular* truss may be used to designate it, distinguishing it from the single quadrangular of a preceding section. Many apply the term *double intersection*, meaning that the tie crosses two panels: it is also known as the Whipple Truss, and is probably better entitled to that name than is the truss of Fig. 11, II, III. An inspection of the figure will show that there are two independent systems of bracing. The two trusses thus formed might, therefore, be analyzed separately, and the chord-stresses found in the complete truss by the addition of the two stresses which would co-exist in each panel. Thus one system would give a certain compression in B F, the other system a compression in D I: the addition of these stresses would give the stress in the piece D F.

49. **Chord-Stresses.** — But it is quite easy to treat the truss as a whole. The bending moments and shears being independ-

ent of the type of truss, the two diagrams for these quantities will be drawn as usual, and are seen below the truss. The ordinates in the first diagram may as well denote chord-stresses for a single truss of ten panels, as the bending moments will all be divided by the constant height A B. One method of analysis will now be given: a second will follow in § 53 (also see § 55). When the truss is fully loaded, the counters, represented by dotted lines, will be free from stress, and the middle ordinate, N′ O′, will be the compression in the top chord from I to S. Upon making a vertical section, and taking moments at K, the stress in I L would be given by the ordinate K′ L′, if it were not for the piece I N, whose stress has a horizontal component, with a moment arm about K equal to one-half the height of the truss; but, as we have found already that the stress in I L is equal to N′ O′, the fact that N′ O′ exceeds K′ L′ must be due to the tension in I N. If, again, moments are taken about L, the stress in K N would also be K′ L′, if it were not for this same piece I N; and, as it is evident that the horizontal component of the stress in I N where it crosses L K acts with only half the arm with which a chord-stress acts, this horizontal component in I N must be double the amount of force, which, exerted in the chord, would have the same moment. And as, at any section in the panel K N, the tension in K N plus the horizontal component of the tension in I N must equal the compression in L O, we will project O′ horizontally to T′, lay off the distance T′ L′ at L′ U′, making T′ U′ = 2 T′ L′; when, T′ U′ being equal to the horizontal component of the stress in I N, K′ U′ is equal to the tension in K N. By construction, K′ U′ + U′ T′ = N′ O′; and, as it were, half the force from I N affects I L, and the other half affects K N. Since the tension in K N equals the compression in F I, by a similar course of reasoning we can get the stress in G K from the stress just found in F I or K N.

But perhaps the reasoning is more clearly stated algebraically. Let C = compression in any piece of the top chord, such as

D F. Let T = tension in piece of the bottom chord E G, on the other side of a vertical section at E. Let H′ = horizontal component of tension in the diagonal D G which crosses the section, and k = height of truss; then, taking moments,

$$\begin{aligned} &\text{About E,} \quad M = C \cdot k - H' \cdot \tfrac{1}{2} k. \\ &\text{About F,} \quad M = T \cdot k + H' \cdot \tfrac{1}{2} k. \\ &\text{Adding,} \quad 2M = (T + C)\, k, \text{ or } T + C = 2M \div k. \end{aligned}$$

As $M \div k$ = ordinate E′ F′, it is evident that T must fall short of E′ F′ by the amount that C exceeds it.

Hence follows the rule: Project O′ horizontally to T′; lay off T′ L′, the intercepted portion of the vertical that is below L′, at L′ U′; project U′ similarly to V′; lay off V′ I′ at I′ W′; and so proceed as indicated in the figure. The ordinates terminating at O′, U′, W′, &c., will be the chord-stresses in I O, F I, D F, &c., while the lower chord-stresses will be the same as in those pieces of the top chord which come between the same neighboring diagonals; or, if preferred, the points marked within the curve belong to the adjoining panels of the lower chord on the side next the middle, and the points without the curve to the adjoining panels of the upper chord on the side of the abutment. See p. 77.

50. **Stresses in Diagonals and Verticals.** — Turning our attention to the shear diagram, we see that the shear in the first panel, A C, is equal to cf, the ordinate below the middle of this panel, and that this shear must be divided between the two ties in this panel, B C having a little the larger portion, as the set of trussing to which it belongs has one more loaded joint than the other system. When B C is strained to its maximum, the load must cover the whole truss, including the joint C: therefore the counters will not be in action, and I N will have a stress due to one-half the load at N. To this vertical component as it passes through G D will be added the load at G, the corresponding load at R being carried to the other abutment; then the load on G D must pass through B C, and, in

addition to it, the load at C, making the vertical component of the tension in B C equal to two and one-half panel weights. Since there are ten panels in the span, this amount is one-quarter of the entire weight of the truss and load. As *a b* represents one-half of the same combined weight, divide *a b* at its middle point *d*, and project *d* horizontally to *u;* then *a d*, *d b*, or *c u*, will be the shear in B C, and *u f* must be left for B E.

When we pass to the next panel C E, the total shear will be *e i*, distributed in B E and D G when the load extends from T to E inclusive. But the removal of the load from C can produce no effect on B E: hence *u f* will still be the shear in B E; and, if *u f* is laid off at *e v*, *v i* will be the shear in D G. Thus all the construction necessary is, bisect the end vertical *a b*, lay off by scale or parallel lines the upper half at the bottom of the next ordinate, then the upper portion of that ordinate at the bottom of the next, and so on, when the two portions of each ordinate will be the shears in the two diagonals in the panel whose middle it occupies, the lower portion belonging to the lower diagonal, and the upper portion to the one above. By drawing lines parallel to the ties, the stresses in them are obtained. The verticals carry the stresses of the diagonals which run to their upper or unloaded ends.

51. **Inclined End-Posts.** — Fig. 21 shows the usual modification of the ends of this truss. Such a change introduces a certain though generally trivial indetermination in the calculation of the stresses: it arises from the fact that the weight at W, conveyed to X by the tension rod W X, may be classed with the system of weights which affect X R, R S, S N, &c., or with X U, U V, V P, &c. If we use the shear diagram of Fig. 20, or imagine the truss of that figure to be modified at the ends, we see that the shear in T X will be *c f*. The total shear in the next panel U W will be *e i*. The rolling load previously at W has been removed for maximum shear in the panel U W, and hence there is a doubt whether its removal has affected X U, X R, or both. If it belongs with the weights at R, N,

&c., the previous analysis for shear is correct, and $e\,v$ will belong to X U, while $v\,i$ will be the shear in X R. If, on the other hand, it belongs with the weights at U, P, &c., X R will have a greater stress, and X U a less, than just given. When the panel weight of rolling load w' was removed from W, the reaction at T was, for this example, diminished by $0.9\,w'$, and at A by $0.1\,w'$: therefore the shear in X U, on the second supposition, will be $e\,v - 0.1\,w'$, and in X R will be $v\,i + 0.1\,w'$. The alternate points v, w, &c., will therefore move down and up by this constant amount; and the indeterminateness lies within the amount $\frac{w'}{N}$, where N = number of panels.

What portion of the shear in X U this change affects may be seen as follows: If w = panel weight of steady load, $b\,d = \frac{1}{2}\,a\,b = c\,u = \frac{1}{4}\,N\,(w + w')$, $c\,f = \frac{1}{2}\,(N - 1)\,(w + w')$, $e\,v = u\,f = c\,f - c\,u = (\frac{1}{4}\,N - \frac{1}{2})\,(w + w')$; then the shear in X U may be altered to $(\frac{1}{4}\,N - \frac{1}{2})\,(w + w') - \frac{w'}{N}$. The variation in amount will not usually be more than one or two tons, changing the cross-section an almost inappreciable amount.

52. **Odd Number of Panels.** — If this type of truss, Fig. 22, has an odd number of panels, the method of finding the stresses in the chords will not be changed; but it will be noticed that K N, the middle panel of the lower chord, has the same amount of stress as exists in F V. By referring to § 43, Fig. 16, the change in the moment or chord-stress diagram will be seen, and the analysis may be conducted as usual. When we turn to the consideration of the stresses in the diagonals, we find that when the whole truss is loaded, and consequently the ties in the middle panel as well as the rest of the counters are not in action, each system of bracing has the same number of loaded joints. The shears in B C and B E in the first panel must then be equal, and each will be one-half of the ordinate for that panel. If the rolling load is removed from C, it first appears that the subtraction of the shear in B E from the ordinate for shear belonging to the second panel will determine the

remainder to be carried by D G; for the system of weights E K N R has not apparently been disturbed. But the removal of the weight from C carries the centre of gravity of the *remaining* weights on the system C G P S to the right of the mid-span, and hence brings into action one of the dotted diagonals. As those diagonals which cross the middle connect loaded joints of alternate systems, the distribution of the shear in the two diagonals of any panel is rendered indeterminate between certain limits.

To illustrate: Remove the load from C, and suppose that the stress in B E is unchanged. The shear in B E will then be $2\,(w + w')$, if w = steady load, and w' = rolling load per joint. The shear in F K will be $w + w'$, and the load at K will then reduce the shear to zero, so that no counter is needed for this system. Passing on, we should have $2\,(w + w')$ in R U also. Again: as the reaction at A must be $4\,w + 3\frac{1}{9}\,w'$, the shear in B C will be $2\,w + 1\frac{1}{9}\,w'$, in D G $w + 1\frac{1}{9}\,w'$; and, on passing G, the shear is $\frac{1}{9}\,w'$, which must pass through I N: then we shall have $w + 1\frac{1}{9}\,w'$ in N V, and $2\,w + 2\frac{1}{9}\,w'$ in R U; which does not agree with the previous deduction. If, on the other hand, we start from T with a reaction of $4\,w + 3\frac{8}{9}\,w'$, and suppose that $2\,(w + w')$ passes through R U, $2\,w + 1\frac{8}{9}\,w'$ is left for U S. The diagonal W P will then carry $w + \frac{8}{9}\,w'$; and, on passing P, the shear changes to $-\frac{1}{9}\,w'$, which must, therefore, go through P L. The shear in F K becomes $w + 1\frac{1}{9}\,w'$, and in B E $2\,w + 2\frac{1}{9}\,w'$, which is greater than before. We may, therefore, have the counter I N or L P in action as soon as a load is removed from C; and B E may, on one supposition. carry more than when C was loaded. The total shear in any panel will remain the same; but the distribution of a small portion between the two ties will be in doubt. An odd number of panels for a double-system truss is not desirable on this account. The two possible ways may be provided for by a sufficient cross-section of the pieces in question.

53. **Triple Quadrangular Truss; Chord-Stresses.** — In this truss, Fig. 23, the diagonals cross three panels. In finding the

chord-stresses here, we will, for practice, avail ourselves of a method which might have been applied to Fig. 20. By reference to that figure, it will be seen, that, the truss being loaded throughout, the difference of the stresses in L O and K N is the horizontal component of the stress in I N, and is, therefore, equal to U′ T′. Similarly, for the panel G K, the difference between K′ T′ in I L and G′ W′ in G K is the sum of the horizontal components in I N and F K: U′ T′ being equal to the former, V′ W′ must be equal to the latter. The vertical component of the stress in I N is one-half the load at N. If the member slopes at 45°, the horizontal component will be the same. The vertical component in F K is the load at K; that in D G is the amount in I N plus the load at G: hence the horizontal components U′ T′, W′ V′, &c., being readily obtained, can be plotted in a series of steps from O′ up, and the desired chord-stresses are obtained.

Turning our attention to Fig 23, neglecting all counters, and proceeding to find the chord-stresses, we will note, that, if w'' denotes a panel weight of steady and moving load combined, the shear in I P $= \frac{1}{2} w''$, in F N $= w''$, in D K $= w''$, in B G $= 1\frac{1}{2} w''$, in B E $= 2 w''$, and in B C $= 2 w''$. If the ties slope at the economical angle 45°, the horizontal components also will be as above; if not, the horizontal component will be the product of the vertical component into the ratio of the horizontal projection of the tie to the height of the truss. Let us suppose that they incline at 45°, with the exception of B E and B C. Determine the middle ordinate P′ Q′ for chord-stress at the middle of the span by the formula of § 28. This stress will be found from I to Q. Draw Q′ L′ horizontally. The stress in N P $=$ P′ Q′ $- \frac{1}{2} w''$. Lay off $\frac{1}{2} w''$ from O′ to O″, and the stress in N P $=$ N′ O″. Stress in N P $=$ stress in F I: therefore project O″ horizontally across two panels to I′, and do the same with points determined later. Imagining an oblique section passed through F I and K N, we see that the difference of stresses in these two chord-pieces is due to F N: therefore stress in K N $=$ stress in F I $- w''$: lay off w'' from the point

where $O'' I'$ crosses $K' L'$, obtaining $K' L''$ as the tension in K N, and compression in $D F = E' F'$. Similarly, stress in $G K =$ stress in $D F - w''$, or $K' L'' - w'' = G' I''$. Stress in $E G =$ stress in G K, or B D, $- 1\frac{1}{2}\, w'' = E' F''$. Stress in $C E =$ stress in $E G -$ horizontal component of stress in $B E = E' F'' - \frac{5}{8}\, 2\, w''$, or $C' D''$, since B E only runs two panels. The accuracy of the results is checked by finding that $C' D'' =$ horizontal component of stress in $B C = \frac{1}{8} . 2w''$. The points O'', L'', &c., within, give stresses on the pieces of lower chord which lie to their right; and the points Q', L', I', &c., without, belong to the pieces of the upper chord which lie to their left.

54. **Stresses in Diagonals and Verticals.** — When this truss is completely loaded, the points G, P, and V belong to one web system; E, N, R, and W, by symmetry of load on the span, belong to a second system; and C, K, T, and X belong to the third web system. If, however, the counters run at the same inclination with the main ties, the latter two systems will be joined to one another by the counters, and the distribution of the shear for a partial load will be rendered indeterminate to a certain small amount; the sum of the shears for the two ties of those systems which cross any panel being a definite quantity. As this matter has been referred to before, we will suggest the modification of the counters shown in the figure; those diagonals which cross the centre running to joints of their own web system, as noted above. The change affects but two sets of counters, and the ambiguity is removed. The truss being so arranged, we may find the web stresses as follows: —

The several systems carry, as seen above, three, four, and four panel weights respectively. The shear in the panel A C, when the entire truss is loaded, will be, for each tie, half the load on its system. Laying these quantities off in order from below upwards at $b\,w$, $w\,x$, and $x\,y$, they will be the shears in B C, B E, and B G in the same order. The diagram is reversed to save room. When the load is removed from C, the shear in the panel C E is $c\,d$: the shears in B E and B G not being affected by the

removal of this load, cut off $w\,y$ from $c\,d$ by drawing a line $y\,k$ parallel to $w\,c$. Draw another parallel line from x: the shear in D K will be $d\,k$. In the same way, upon the removal of the load at E, the shear in the panel E G will be $e\,f$; and upon drawing $d\,n$ parallel to $z\,e$, making $n\,e = d\,z$, the shear $f\,n$ in F N is found. Thus we proceed until we reach the panel P R. Here it will be seen that the counters O R and L T have changed their relative positions in crossing the middle vertical. As, however, $o\,r$ and $l\,t$ are to be subtracted from the ordinate at q, the usual construction will be correct. In the panel R T are found the ties L T, Q V, and S W, if necessary: therefore $l\,t$ and $q\,v$ must be subtracted from the ordinate at s; and, as they do not come on consecutive ordinates, they will be subtracted separately. They exceed the ordinate at s; but, if any positive shear in other examples exists beyond this panel, the previous method of finding the remainder may be resumed.

55. **Double Triangular or Warren Truss.** — If the truss of Fig. 24 is taken as a whole, we may discuss it by either of the methods employed with the Quadrangular Truss. We will pursue one method, and leave the other for the reader to follow out if he pleases. When the truss is fully loaded, the tie L N, which runs from the middle loaded joint N, will have a shear of one-half a panel weight. This shear will also be found in the strut L G; and, after passing the point G, it will be increased by the additional weight at that point, the shear in F G and F C being one panel weight and a half. In the other system the central trapezoid I K P S is in equilibrium under the weights at K and P; so that K O and O P have no stress, and the shear in I K and I E is one panel weight. In D E and D A there will be a shear of two weights. If we draw the usual polygon for chord-stresses, A′ D′ F′ O′, &c., we see, upon taking moments about N, that N′ O′ is the stress in L Q. If a section is made at L, and moments taken at that point, we also find, since K O has no stress, that the tension in K N is K′ L′. The difference between K′ L′ and N′ O′, or O′ U, must be the horizontal component of the stress in L N, or the horizontal component

due to a shear of half a panel weight. As all the braces, struts or ties, slope at the same angle, the horizontal components of their stresses will be simple multiples of O′ U. The difference between the stress in L O and that in I L will be due to the two horizontal components in L N and L G; likewise, in the bottom chord, the stress in E G will be less than that in G K by the sum of the horizontal components in L G and F G. The compression in L O having been determined as N′ O′, and the horizontal component of the stress in L N having proved to be O′ U, subtract 2 O′ U from N′ O′, or make L′ L″ = O′ U, to obtain K′ L″, the compression in I L. The stress in F I will again be diminished by reason of the action of I K and I E, or by 4 O′ U. From the well-known property of the parabola, if O′ U is the distance that L′ is vertically above O′, I′ is four times that distance above O′. As twice O′ U has been subtracted already from N′ O′, to subtract 6 O′ U it is necessary to make I′ I″ = 2 O′ U, and G′ I″ will be the compression in F I. To get the stress in D F we must again subtract 6 O′ U, or 12 O′ U from O′ N′. From O′ to F′ vertically being 9 O′ U, make F′ F″ = 3 O′ U, and E′ F″ will be the compression in D F. In the same way, 8 O′ U must be again deducted upon passing the joint D, or D′ D″ = 4 O′ U.

For the bottom chord the stress in K N was found to be K′ L′. The tension in G K will be less by the horizontal component of I K or 2 O′ U. As O′ U has been subtracted in going from O′ vertically to L′, and as the whole vertical distance to I′ is 4 O′ U, to subtract, in all, 3 O′ U from N′ O′, brings us to I‴, a distance I′ I‴ = O′ U, below I′. Passing G, we must subtract 4 O′ U, or, in short, add 2 O′ U to E′ F′, making E′ F‴ = tension in E G. Without elaborating further, we have the rule:—

Set off, inside the parabola, once, twice, thrice, &c., the distance which O′ N′ exceeds K′ L′ on the successive ordinates from the middle, at L″, I″, F″, &c., and outside the parabola, at I‴, F‴, D‴, and A‴, commencing on the second ordinate from the middle. The inside points will determine the

upper chord-stresses, and the outside points the lower chord-stresses. The accuracy of the construction will be checked by $C' D''$ and $A' A'''$, being respectively the horizontal components of the stresses in B C and A D. The construction on the right half of the chord-stress diagram of Fig. 20 may now be proved for the truss of that figure, the difference between $N' O'$ and $K' L'$ being laid off once, twice, thrice, &c., on successive ordinates, both inside and outside of the curve, as indicated by the small circles.

The construction of Fig. 24 may also be made as follows: Draw $L' U$, $I' V$, $F' W$, &c., horizontally across one panel; lay off $O' U$ at $L' L''$ and $I' I'''$; lay off $L'' V$ at $I' I''$ and $F' F'''$; and so proceed to the close.

The distribution of the maximum shear between the two diagonals of each panel is easily obtained. One of the two systems into which the web members may be divided, having one more loaded joint than has the other system, will carry one-half the entire weight of one truss. For example, the truss of Fig. 24 has ten panels; the system B C F G L N, &c., has five loads: hence, when the truss is fully loaded, the shear in B C will be one-half of the *end* ordinate; and, if this half is projected upon the ordinate in the middle of the first panel, the remaining portion will be the shear in A D. If the load is withdrawn from C only, the shear in D E, which belongs to the system not connected with C, cannot be disturbed, and must be equal to the amount just obtained in A D. Deduct this amount from the ordinate for the panel C E, and the remainder will be the shear in C F. Thus the operation is the same as in finding the vertical components of the stresses in the diagonals of the double quadrangular truss, § 50. Some pieces in the middle portion of the span will be subjected to alternating stresses of tension and compression, as in the single triangular truss.

If the load is upon the upper chord, the chord-stresses will change from one chord to the other, and the diagonal stresses from one diagonal to the other, all in the same panel; the

change being one of amount and distribution, not of kind. That the stresses are changed arises from the fact that a load on a joint, when shifted from one chord to the other, is also moved laterally in its own system half a panel: in other words, the two systems change places.

56. **Lattice Girders.** — Multiple systems are used in the triangular type of truss in riveted bridges, when the diagonals are frequently riveted together at intersections, and a *lattice girder* is produced. It is probable that such connection of the diagonals causes more or less distribution of stress from one web member to another; and the compression members, being stayed at frequent intervals, may be much more slender than would otherwise be possible. Although the several systems of Fig. 25 may be distinguished, and the stresses carefully determined, riveting of the diagonals to one another destroys the accuracy of such discrimination; and hence the assumption which is commonly made — that the shear at any section will be equally distributed over the pieces of the web cut by the section, and that the chord-stress in any panel will be found by dividing the bending moment at the middle of that panel by the height of truss — does not materially err from the truth.

For while it will be noticed, that, in such a truss as the quadrangular, the stresses in the two opposite chord-pieces of the same panel differ materially, — in fact, by the amount of the horizontal components of the web members in the panel, — it will be seen, that in the double triangular truss, and also in the lattice girder, the horizontal components of the stresses in the strut and tie diagonals in any panel partially neutralize one another, thus bringing the opposite chord-stresses more nearly to an equality. Thus in Fig. 24 the compression in F I is G′ I″, and the tension in E G is E′ F‴. They do not differ greatly from one another, nor from the ordinate at the middle of the panel: hence the above rule will give a very tolerable result.

The verticals shown in the figure are introduced at more or less frequent intervals, partly to distribute the moving load

between the upper and lower chords, but more particularly to form, with the struts of the *lateral* bracing at top and bottom, stiff frames to resist lateral distortion. Gusset-pieces also may be riveted in the interior angles between these verticals and the lateral struts.

57. **Post Truss.** — The type of truss illustrated by Figs. 26, 28, and 29, takes its name from its inventor: it is probably the outgrowth of an endeavor to study economy of material by inclining the struts and ties at the most economical angle. While the length of a tie has no influence on the cross-section necessary to carry safely a certain stress, the dimensions of a strut depend upon its length; and hence, the shorter it can be made, the less its cross-section will be. Mr. S. S. Post proposed, that, calling the distance between loaded points one panel, a tie should extend across one panel and a half, and a strut across one-half of a panel, the height of the truss being one panel and a half; so that the ties should slope at an angle of 45°, and the struts at an angle of 18° 26′, with the vertical. As a tie and a strut together extend over two panels, the web members must form a double system. The counter-ties will also slope at 45°, owing to the peculiar arrangement of parts. The joints in one chord occur midway between those of the other; and it will be noticed that each counter connects the joint of one system with a joint of the other system, thus introducing an element of indetermination into the attempt to find the web-stresses.

While this form of truss has been, so far as our knowledge goes, used exclusively as an over-grade or through truss, there appears to be no reason why it may not be employed in a design for a deck-bridge; and, as the method of finding the chord-stresses differs in the two cases, we shall give both.

58. **Load on Bottom Chord; Chord-Stresses.** — When the moving load is carried upon the bottom chord, we notice a peculiarity in the distribution of this load on the truss, which at once distinguishes the latter from all that have preceded. If the truss is N panels long, N loads are concentrated at joints

of the lower chord, the first and last loads being distant half a panel from the abutments; so that, if the span is divided into N equal portions, the loads are concentrated at the middle of each portion, instead of at the points of division as before. The bending moment at the middle is not affected; and hence the chord-stresses at the middle are unchanged by such an arrangement. If $w'' =$ a panel weight of steady and moving load, $l =$ span of truss, and $N =$ number of loads and of panels in top chord, the bending moment at mid-span will be

$$M = \tfrac{1}{2} N w'' . \tfrac{1}{2} l - \tfrac{1}{2} N w'' . \tfrac{1}{4} l = \tfrac{1}{8} N w'' l,$$

as formerly shown, § 28. If the load on the first joint from the abutment is considered to be only $\frac{3}{4} w''$, the value of M will be slightly less.

The equilibrium polygon can be drawn as usual, its vertices coming upon the verticals let fall from the loaded points; but it must be remembered that no half-weight is carried upon the abutment, and that, therefore, the whole load-line is to be included between the lines which radiate from the pole. It may not be amiss to show the relation of this equilibrium poly gon to the parabola for uniform load over the entire span and to the polygon previously described. If A D, Fig. 27, represents a portion of a beam uniformly loaded per horizontal foot, the equilibrium curve for such a load will be the dotted parabola A′ B′ C′ D′. If the same load is concentrated at A, B, C, D, &c., the parabola changes to the equilibrium polygon A′ B′ C′ D′, the removal of the load over A E to the point of support A permitting the portion A′ B′ to rise from the curve to the straight line; but the points A′, B′, C′, and D′, correspond in the two cases. If, on the other hand, the load is concentrated at E, F, G, &c., midway between A B, B C, C D, by the principle explained in § 10, the lines B′ F′ and C′ F′, tangents to the parabola at B′ and C′, must intersect at F′ in the vertical through the centre of gravity of the load on B C; and hence A′ E′ F′ G′, &c., will be the new equilibrium polygon *tangent* to the parabola at A′, B′, C′, and D′. The three figures have these

points in common; and, having one equilibrium polygon or curve, we can readily construct the others. If I, K, and L denote the points where the straight lines $A'B'$, $B'C'$, &c., cut the verticals through E', F', and G', it may be noticed, for convenience, that the parabola bisects IE', KF', &c.; that $IE' = KF' = LG'$, &c., and that $E'F'$ is parallel to the chord which may be drawn from A' to C'; that $F'G'$ is parallel to a chord from B' to D', &c. Some of these facts will be made use of at once.

Let the half-span of the Post Truss, loaded on the lower chord, be represented by A P, Fig. 28. As the load is complete and symmetrical, there will be no stress in A B; and the stress in A C, as well as in the piece to the left of B, will be given by dividing the formula of this section by the height of the truss, one panel and a half. Lay off the quotient at $U'A'$. As we are most familiar with the interior equilibrium polygon of Fig. 27, drop dotted verticals from C, E, G, &c., and construct the dotted polygon by the method of § 28, Fig. 8. The tie which runs from B to E transfers w'' to the abutment Q as a vertical component: since it slopes at 45°, the horizontal component of its stress will also be w''. In the same way, the horizontal component of the stress in D G will be w''; of that in F K, and of that in I N, $2\,w''$; and so on. The struts which meet these ties at their upper ends, having only one-third the inclination from the vertical of the ties, will have stresses whose horizontal components are one-third of those belonging to their respective ties: hence the horizontal component from A B and C D will be zero; from E F and G I $= \frac{1}{3}\,w''$; from K L and N O $= \frac{2}{3}\,w''$, &c. It will be noticed, then, that the *increments* of stress in any chord from the abutment to the middle will be in pairs: for example, the stress in G K will exceed the stress in K N by the same amount that the stress in K N exceeds that in N P.

The middle ordinate $U'A'$ is $\dfrac{w''\,N\,l}{8 \cdot PQ} = \dfrac{w''\,N^2}{12}$, since $PQ = \frac{3}{2}\,\dfrac{l}{N}$. If $A'C'$, drawn horizontally, were continued to the

right, the successive vertices of the dotted polygon would, since they lie on a parabola, be vertically above A′ C′ by amounts varying as the square of their horizontal distances from A′. The vertical distance C′ V will, therefore, be $\frac{w'' N^2}{12} \div \left(\frac{N}{2}\right)^2 = \frac{1}{3} w''$. The next vertex, at E′, will be $\frac{4}{3} w''$ above A′, the next $\frac{9}{3} w''$, and so on. If verticals are let drop from B, D, F, they will cut the polygon already drawn in the middle of the respective sides; and, if desired, the external polygon of Fig. 27 can be at once drawn. Computing a few of the chord-stresses, beginning at the middle, we see, that, in A E, there is compression of $A' U' = C' V' = \frac{w'' N^2}{12}$: for E G we subtract $1\frac{1}{3} w''$, arising from B E and E F; for G K, $2\frac{2}{3} w''$, or $1\frac{1}{3} w''$ additional from D G and G I; for K N, $2\frac{2}{3} w''$ additional, or $\frac{16}{3} w''$; and so on. In the bottom chord the tension in B D is less than A′ U′ by w''; it again diminishes in D F by w'', in F I by $2\frac{1}{3} w''$, &c. If the horizontal components of the diagonal stresses are marked on the inclined pieces, the changes are readily noted. It will now be seen that the decrease in the values of the ordinates to the dotted polygon at the alternate joints A′, E′, K′, P′, &c., corresponds to the decrease in the top chord-stresses in the alternate pieces A C, E G, K N, &c.; and that the bottom chord-stresses in D F, I L, &c., are *means* of the stresses in the two pieces of the top chord respectively, E G and G K, K N and N P, &c.: hence, when we remember that the increments in each chord go in pairs, we get the following construction: —

Draw A′ C′ horizontally; through C′ and E′ draw C′ G′; through G′ and K′ draw G′ N′, &c. The points where this polygon cuts the verticals through B, F, L, being marked, draw the polygon B′ F′ L′, &c. The points where the *exterior* polygon cuts the verticals from the top chord joints will give the stresses in the successive pieces of the *top chord* on the *right* of the joints, and the points where the *interior* polygon cuts the verticals from the bottom chord joints will give the *bottom*

chord-stresses on the *left* of the respective joints. For accuracy of construction, it may be noticed that the distance V C′ may be set off at W G′, X N′, &c., or C′ G′ drawn parallel to a line from A′ to K′, &c. The construction is much more simple and brief than is its analysis. The line L′ O′ does not pass through P′; but O′ is above N′ P′ by the same amount that D′ is above C′ E′.

59. **Load on Top Chord; Chord-Stresses.**—In this case, represented by Fig. 29, the horizontal component of the stress in A B for a complete load will be $\frac{1}{6}\,w''$, in C D $= \frac{1}{3}\,w''$, in E F $= \frac{1}{2}\,w''$, &c.; in B E $= \frac{1}{2}\,w''$, in D G $= w''$, in F K $= 1\frac{1}{2}\,w''$, &c. The chord-stress in the middle piece of the bottom chord will be A′ U′, the same amount as before, and as usual; the stress in B D will be less than that at the middle by $\frac{2}{3}\,w''$; that in D F will decrease $1\frac{1}{3}\,w''$, or will be $2\,w''$ less than the stress at the middle; and so on. If we draw the usual equilibrium polygon, shown by the dotted lines, on the verticals from the loaded points, and then construct A′ B′ D′ F′, &c., as the exterior polygon of Fig. 27, by either of the methods given, we shall find that the points B′, D′, F′, &c., satisfy the values for the pieces of the *bottom* chord to their left; for it was shown in the last section that the successive vertices of the first polygon came above the horizontal line A′ B′ by $\frac{1}{3}\,w''$, $\frac{4}{3}\,w''$, $\frac{9}{3}\,w''$, &c: the point D′ will, therefore, be $\frac{2}{3}\,w''$ vertically above B′; F′ will be $\frac{6}{3}\,w''$ above the same point; and so on. In the top chord the stress in A C will be $\frac{1}{6}\,w''$ less than A′ U′; in C E it will be $\frac{1}{2}\,w''$ less; in E G, w'' less than A′ U′; and so on. These stresses are the means of those cut off from the verticals at the lower chord joints by the first polygon at V, W, X, &c.: hence, if we draw V W X Y, &c., the points C′, E′, G′, &c., where this figure cuts the top chord verticals, will determine the stresses in the pieces of top chord on the right of each vertical. As B′ V, D′ W, F′ X, are all equal, V W is parallel to B′ D′, W X to D′ F′, &c., and the points C′, E′, &c., are at a constant vertical distance within the exterior polygon: as this distance is B′ V, it can be at once set off at the alternate verticals.

60. **Web-Stresses.** — When the load is upon the bottom chord, and extends from end to end of the truss, all of the web-pieces which meet at the middle upper joint, as well as those which cross the centre line, will be free from stress. The end vertical of the shear diagram will equal the weight upon half of the truss, and will be divided into two portions equal to the amount from each system carried by that abutment. In our figure it is to be bisected: if the number of loaded joints in the half-span is odd, one of the portions will exceed the other by w''. If the shear in P L as thus obtained is deducted from the ordinate under the middle of the panel O L, the remaining shear will be found in O N and N I upon the removal of the rolling load from O, the symmetry of the loading of the system which includes L not being disturbed as yet. But the removal of the load from L, also, leaves loads on each system whose centres of gravity lie a little to the left of the middle of the span; and hence some of the braces and counters which cross the centre line, or meet at A, will be called into action. So many paths are open to the shear at that section, that the lines over which it will pass cannot be definitely ascertained. It is probable that the easiest and safest way will be to make a shear diagram like that of Fig. 20, and to take the larger portion of each vertical ordinate as the possible shear in the diagonals of that panel to which the vertical belongs. Similar remarks apply to Fig. 29, the shear verticals being drawn under the middle of each joint of the bottom chord, and the vertical at the middle of the first panel being divided as above described, and as was done in Fig. 23 for three systems.

61. **Effect of Locomotive on Double System.** — The diagrams for the extra weight of the locomotive above the average for the train may be added without difficulty to those for trusses with double or multiple systems of bracing. The shear in any panel from this extra weight is properly considered as coming upon that diagonal which runs to the loaded joint, as this load, unless it covers more than one panel, will not strain both systems at once. Where the panel joints are so near

together that two or more are loaded at once, an inspection of Fig. 10 will show that the maximum ordinate at each joint in succession will belong to the diagonal which sustains that joint, although this maximum is somewhat less than the other.

Short Construction for Chord-Stresses.—If a straight line is drawn from N′, Fig. 20, to a point, on the vertical below A′, which corresponds with K in Fig. 8, the lower chord-stresses may at once be measured from the equilibrium polygon to this line; and a prolongation of this line to the right of N′ will give the upper chord-stresses.

A similar line drawn from N′, in Fig. 24, to a point below A′, will cut off the ordinates for upper chord-stresses in the double triangular, or Warren truss, while a line drawn from K to the left at the same rate of inclination, but sloping the opposite way, will limit the ordinates for lower chord-stresses. If the load is changed to the upper chord, the same stresses will be found as before, but in each instance in the other chord.

CHAPTER V.

TRUSSES WITH INCLINED CHORDS.

62. **General Remarks ; Effect on Chord-Stresses.** — While the determination of the stresses in a truss with inclined or curved chords is not quite so simple a matter as where the chords are parallel, no particular difficulty will be experienced, if, as with parallel chords, the members are considered to be jointed at each intersection. Take, for example, the truss represented by A B C D, Fig. 30. The span and the load are the same as those of the truss, Fig. 6, before described. The bending moment at each joint, and the shear in each panel, for the truss as a whole, will be unaltered, and the diagrams of Fig. 6, therefore, will apply here. In many practical examples one chord or the other is made straight throughout its entire length; but both are inclined in this case for a general illustration.

The maximum bending moment at any joint will then be obtained by multiplying the proper ordinate of the equilibrium polygon, A′ F′ I′ K′ D′, by H from the stress diagram; and this moment must be equal, by the theorem of moments and by what has been previously said, to the stress in the proper chord-piece multiplied by the *perpendicular* from the origin of moments to that piece of the chord. As this perpendicular is not always a convenient quantity to use, and will not be the same for the two chord-pieces in the same panel, it probably will be better to substitute for the product just mentioned the equal product of the horizontal component of the stress in the chord multiplied by the height of the truss at the joint in question. That one product is equal to the other is easily seen, if

we notice that the forces and lines referred to are respectively base and hypothenuse of two right triangles whose sides are perpendicular to one another. From the horizontal component may be obtained the stress itself by multiplying the former by the ratio of the length of the inclined piece to its horizontal projection; that is, by multiplying by the length of the inclined chord in the panel, and dividing by the horizontal distance between panel joints. If we call the length of a panel horizontally a, and the difference of level of the two joints in the same chord b, we may multiply the horizontal component above by

$$\sqrt{\left(1+\frac{b^2}{a^2}\right)},$$

and we shall obtain the direct stress in that piece of the chord: hence, *to find the stress in any piece of the chord*, multiply the proper ordinate under one end of the piece by H, divide by the height of the truss at that joint, and multiply by the above expression.

63. **Effect on Stresses in Diagonals.** — As one or both of the chords are inclined, such inclined members are able to and must convey a certain portion of the vertical force which exists at a section in any panel. Thus, suppose a vertical plane of section to be passed through the panel R F G Q of Fig. 30. Neglecting one of the diagonals, as we know that in a good design both cannot be strained at once (see § 23, last part), the vertical shear at the section is distributed over the three pieces R Q, R G, and F G, all of which are inclined, and therefore capable of carrying a vertical component. As the portions of vertical force in R Q and F G will be just sufficient to cause, when combined with the horizontal components derived from bending moments, *direct* stresses along those chord-pieces, it will only be necessary to find those horizontal components for such a distribution of load as gives maximum shear in the panel, to easily deduce the amount of shear which passes through the chords: the balance of the shear will be left for the diagonal to carry.

For example: To find the maximum stress in any diagonal, such as R G, the rolling load will be moved so as to cover all joints from D to G inclusive (see § 69). The equilibrium polygon, reproduced from Fig. 6, is D′ K′ I′ F′ A″, and the shear ordinate is $q f$. By inspection of the enlarged sketch of this panel, shown on the right, we see that on one side of the plane of section, say the left hand, we have the vertical shearing force, acting upwards, the horizontal pull, obtained by taking moments at R, which acts through F, and the horizontal thrust, obtained by taking moments at G, which would act through Q. On the other side of the plane of section we find three inclined pieces, whose stresses must balance the rectangular components on the left side.

From the equilibrium polygon for the given position of the load take G′ Q′, multiply by H, divide by G Q, and lay off the quotient, which is the horizontal force through Q, horizontally at $f y$, Diagram I.[1] In the same way, F′ R′ . H, divided by F R, gives us the horizontal force through F, laid off at $f x$. The vertical shearing force, $f q$, is plotted vertically upwards at $f q$. Now draw $q r$ from q, parallel to Q R, and limit it by a vertical from y. Draw $f g$ from f, parallel to F G, till it meets a vertical through x. Draw the line $r g$, and it must be parallel to R G, or some error in construction has been made. Thus we have a check on the accuracy of our work. By dispensing with this check it is necessary to determine only one of the horizontal stresses, and then we may draw the other lines parallel to the respective pieces.

64. **Stresses in the Verticals.** — It remains to find the stress in R F, the vertical which joins the unloaded end of the diagonal. The vertical component in G R will not pass unchanged in amount through R F; for R S and R F together will carry the vertical components of the stresses in Q R and G R. The portion that will pass through R S is definitely fixed by the consideration, that, when combined with the horizontal force at

[1] The scale of these diagrams is increased one-half for distinctness.

the joint R due to bending moment, the resultant must be a direct thrust along R S. It is, however, unnecessary to have recourse anew to the equilibrium polygon. Consider the joint R. This joint is in equilibrium under the action of four forces which meet at that point; and two of them, $q\,r$ and $r\,g$, have just been determined. The two remaining sides of the polygon of force are readily drawn. Q R and R G being taken in order, draw $g\,t$ parallel to F R, and $t\,q$ parallel to R S, to close on q. The required stress in R F is, therefore, the compression $g\,t$. The arrows on this quadrilateral proceed round the figure as usual, and show the directions of the stresses exerted on R by the several pieces.

65. **Remarks.** — It may not be amiss to call attention to the fact, that, since we may dispense with one of our data, that one may be the shear, and thus we may do without the shear diagram; for it will be noticed that the horizontal projection of $r\,g$ is $y\,x$, the difference of the horizontal components in the two pieces of the chords in this panel: hence, by drawing a line from x parallel to R G, we may find the stress in R G very quickly. But, as we then have no check on the correctness of $f\,x$ and $f\,y$, it is hardly advisable to take this course.

The arrows on $q\,r$, $r\,g$, and $g\,f$, show that the three stresses just determined will together balance the shear, since their vertical projection is equal and opposite to it; and that the horizontal projection of the stress in the top chord is exactly balanced by the horizontal projections of the tensions in the tie and lower chord. The horizontal projection of the stress in R S is the same as of that in F G; which result is to be expected, from the fact that these two chord-pieces lie between adjacent diagonals sloping the same way. The stresses $q\,r$, $q\,t$, and $f\,g$, are those existing, in the pieces to which they refer, for the present load; but, as they are not the greatest stresses which the chord-pieces must resist, they are of no special value here. The maximum stresses were determined in § 62.

66. **Construction for Horizontal Components.** — The por-

tion of the figure within the triangle $f\,d\,y$ gives a geometrical construction for finding the horizontal components $f\,x$ and $f\,y$. For example, since

$$f\,y = \frac{G'\,Q' \cdot H}{G\,Q},$$

draw $f\,d$ at any convenient angle with the horizontal line; make $f\,w = H$, $f\,b = G\,Q$, $f\,d = G'\,Q'$; draw $b\,w$, and, parallel to it, draw $d\,y$ through d. The desired component will be $f\,y$; for, from similar triangles,

$$f\,b : H = f\,d : f\,y = \frac{f\,d \cdot H}{f\,b}.$$

Similarly, $f\,a = F\,R$, $f\,c = F'\,R'$, and $c\,x$, drawn parallel to $a\,w$, determines $f\,x$. If $c\,x$ and $d\,y$ cut the horizontal line at favorable angles, the values of $f\,x$ and $f\,y$ may be thus obtained quite satisfactorily.

67. **Stresses when Shorter Segment is loaded.** — The stresses in G Q and Q I, as well as in the panels to the left of the one just discussed, will be obtained by a figure similar to the one already described. After passing the middle of the span, for example, to the panel P I J O, the moving load now extending from D to J inclusive, we shall have Diagram II.; in which case we use ordinates I′ P′ and J′ O′. In this case, $i\,j$, the stress in the lower chord, runs below the horizontal line; but the method of construction is still the same. It is noticeable that the inclinations of the chord-pieces I J and P O will *increase* the vertical force transmitted by the diagonal J P *towards* the abutment A, while the *resultant shear* is unaffected: hence it appears that the stress in any diagonal is *increased* by such an inclination of either chord as tends to make the *height* of the panel, on that side to which the diagonal conveys its load, greater, and *vice versa*.

After we pass the point where the curve which limits the ordinates for shear passes below the horizontal line $b\,c$, the vertical for shearing force must be drawn *below* the horizontal line of these minor stress diagrams. Thus, taking the panel

O J K N, we draw, in Diagram III., *n o*, *o k*, and *k j*, exactly as before. Since *n o* and *k j* cross, we still have tension in the diagonal O K, as shown by the arrows; and we may also find a compression in O J, in case the inclination of O P does not take all of the remaining vertical force, or, as seen by inspection of Diagram III., if the point *v*, on *n v*, does not fall at or below *k*. If it does fall below *k*, there will be tension in the vertical. If we advance one panel nearer abutment D, the moving load now resting on D and L only, the panel N K L M will give us Diagram IV. On attempting to find the stress on a diagonal from N to L, we find, by the necessary direction of the arrow on *n l*, that a piece N L would thrust against the plane of section. As the diagonals are in this example supposed to be ties, we have passed the limit where those sloping in this direction are required. A similar set, from D to F, will complete the truss. If the truss is deeper at the centre than at the ends, the effect of the inclination of the chords is to require more diagonals sloping one way than is necessary for a truss with parallel chords.

The several diagrams for stresses in the web members are readily combined into one, as is done in Fig. 31, the scale of which is also increased one-half: the vertical line contains the shear ordinates all laid off from one point *b*. One additional simplification, explained in the next section, condenses the analysis of trusses with inclined chords into a very brief construction.

68. **Maximum Equilibrium Polygon Sufficient.** — It is not necessary to draw the several equilibrium polygons for rolling loads covering different portions of the truss when we desire to find the stresses on the successive braces. It will be seen, that, when the stress upon the diagonal R G of Fig. 30 is sought, the ordinates F′ R′ and G′ Q′ are used, and that these ordinates, although measured to the polygon which terminates at A″, have their lower extremities at F′ and G′, points which belong as well to the polygon D′ I′ A′. Their upper ends are situated upon the line A″ D′, which closes the polygon D′ I′ A″.

In the same way, for the panel P I J O, the ordinates $I' P'$ and $J' O'$ are included between $D' A'''$ and the main polygon $A' I' D'$. As all the ordinates will be found to terminate on the maximum equilibrium polygon, we need draw no other, if we can locate $D' A''$, $D' A'''$, &c., by finding the points where these lines cut the vertical dropped from A'.

By referring to § 10 we see that the length of an ordinate intercepted between the prolongations of any two lines of the equilibrium polygon will be proportional to the bending moment at that point due to the weights included between the prolonged sides. If, then, a weight is removed from E, Fig. 30, the polygon strikes the vertical below A at A_1. The change in the bending moment at any point of the span, due to the diminution of the supporting force at D, must be equal to the product of H by the ordinate between $D' A'$ and $D' A_1$; and this decrease of moment is all that will be found until we pass the whole of the fully loaded portion. The change due to the removal of the load from E will be proportional to the ordinate between $E' A'$ and $E' A_1$: therefore the intercept $A' A_1$ must be equal to the decrease of reaction at D, multiplied by the span, and divided by H. If $w' =$ rolling load at one panel joint, $N =$ whole number of panels, and $l =$ span of truss, the decrease of reaction at D, due to the removal of w' from the first joint beyond A, is $\frac{w'}{N}$, and the moment of this force about A is $\frac{w' l}{N}$: whence the intercept $A' A_1$ must be $\frac{w' l}{H \cdot N}$.

If the rolling load is next removed from F, the reaction at D will be diminished again by twice the previous amount, and consequently the distance from A_1 to A'' will be $2 \frac{w' l}{H \cdot N}$; the next interval will be three times the above amount; and so on. Hence, if we calculate the above quantity, and lay off in succession, from A', once, twice, three times, &c., that amount, or measure from A' one, three, six, ten, &c., times $\frac{w' l}{H \cdot N}$, the points

so determined will be the extremities of the closing lines; and these lines, with the maximum equilibrium polygon, will supply all of the needed ordinates. They have been added to the polygon of Fig. 30, and the points to which ordinates are measured are marked by small circles. They go in pairs, the two on any line, on each side of a panel, being the ones applicable to that panel.

69. **Rolling Load to extend to the Panel for Maximum Stress in the Brace.** — The stress on any diagonal, such as R G, will be a maximum when the rolling load extends from the abutment D up to the panel, including the joint G, as has been previously proved for trusses with parallel chords. If the load advanced to F, the shear, fq, of Diagram I., would be diminished, the bending moments, and consequently the distances fx and fy, increased, and therefore rg would be shortened. From another point of view, since the equilibrium polygon for a single load is a triangle with the apex under the weight, if a load is added at F, the bending moment at that point will be increased more than will the moment at G, and hence fx will be lengthened more than the increase of fy: hence their difference xy will be diminished, and rg must be less. Again: if the load is withdrawn, so that G is uncovered, the bending moment at G will be decreased more than that at F; so that fy will be diminished more than fx will be: hence xy will again be shortened.

70. **Strut Diagonals; Load on Top Chord.** — If the diagonals had been struts in place of ties, the diagrams would have been constructed with the same ease. In the panel R F G Q, Q F would then be the member whose stress was desired: moments at Q would give the horizontal component of F G; and moments at F, the horizontal component of R Q. The vertical through x would limit qr, and that through y would intercept fg. The diagonal would therefore lie in the other direction, and give a compression. The magnitude of the stress would not necessarily be the same as rg. The tension in the vertical G Q, running to the unloaded end of Q F, would

then be found as usual for four forces in equilibrium at the joint Q.

If the load for this truss were on the top chord, instead of the bottom as here, in place of finding the tension and compression in R G and R F, we should find them in R G and G Q; those two web members connecting two adjacent loaded points. The four pieces which meet at G would then give the desired closed quadrilateral, and the change in Diagram I. would be, that, in place of drawing *q t* and *g t*, we should prolong *f g* (shown by a dotted line) to meet the vertical through *y*, and the upper intercepted portion of this vertical would be the stress in G Q. It is well to notice, that, even with tension diagonals, some one or more of the verticals may occasionally prove to be under tension in some types of truss. Such pieces must be adapted to both kinds of stress. When the stresses on the web members are under investigation, it may be found convenient to use Bow's method of notation, explained in Part I., "Roofs."

71. **Bowstring Girder.** — While the general treatment of trusses with inclined chords, which has now been given, will enable one to analyze any single-span truss of this type, there is one form which deserves special treatment, both from the frequency of its occurrence, and from our ability to develop certain ways of shortening the analysis very materially. The type to which we refer is the bowstring girder. In practical construction the bow is sometimes, perhaps often, bent to an arc of a circle, or the upper ends of the verticals lie upon such a curve, while the several panel lengths of the upper member are straight; but in theoretical treatment, and in many structures, the bow is a parabola, or a polygon coinciding with the equilibrium polygon for a complete load. The parabolic girder will now be discussed, and the circular segment will be referred to later. It may not be amiss to say, that, when the rise of the circular arc does not exceed one-tenth of the span, no error of consequence is committed by assuming the curve to be a parabola.

In the same way,

72. **Chord-Stresses.** — The truss is represented by Fig. 32. The first step will be to find the chord-stresses under a full load. If we draw the equilibrium polygon for such a load, and remember, that, as explained in § 28, the vertices of this polygon all lie in a parabola, we shall see, that when we seek to find the horizontal force at any joint, by multiplying the ordinate to the equilibrium polygon by H, and dividing by the height of the truss at the joint, the ratio of the ordinate divided by the corresponding height of the truss just above it will be a constant quantity; that is, any ordinate divided by the height at the joint in question will be the same as the middle ordinate divided by the centre height. Hence *the horizontal stress will be constant in all the chord-pieces*, the lower chord will have a uniform tension throughout, and in the upper chord the direct stress will increase, from the middle toward each abutment, according to the inclination.

We may find the stresses in the chords graphically as follows: Conceive that the equilibrium polygon passes through the upper ends of the verticals of the truss; then the ratio just referred to is unity, and the stress in the lower chord is H of the stress diagram. As the *equilibrium curve* for a load of uniform intensity over the whole span is a parabola which will pass through the vertices of the equilibrium polygon for concentrated loads, and as the tangent at the springing point of this parabola will cut the middle ordinate at twice the height of the curve above the base, make D J = D C, and draw A J, which, being the tangent, is the direction of the force at the beginning of the equilibrium curve. Lay off half the weight of truss, and complete rolling load at 3–2; draw 2–0 parallel to A J; and 3–0 will be the desired value of H, the stress in the bottom chord, and the horizontal component of the stress in all parts of the bow.

The first side of the polygon for concentrated loads will run from A to E, and will be parallel to a line, which, starting from 0, cuts off below 2 a distance equal to the half-load at A. If, then, 0–4, 0–5, 0–6, &c., are drawn parallel to A E, E F, F G,

G I, and I D, these lines will be the compressions in the respective parts of the top chord, as they will all have a horizontal component H.

The amount of H may be seen from the figure to be, if $2\text{–}3 = \frac{1}{2}\,W''$, $A\,B = l$, and $C\,D = k$,

$$2\,k : \tfrac{1}{2}\,l = \tfrac{1}{2}\,W'' : H = \frac{W''\,l}{8\,k};$$

which agrees with the value deduced in § 28. If the pieces of the top chord are curved, they will be exposed to a small bending moment, equal at their middle points to the direct thrust multiplied by the perpendicular from the chord of the curved piece to its centre line.

73. **No Stress in Diagonals for a Complete Load; Tension in Verticals.** — Since the equilibrium polygon may be drawn to coincide with the bow, for a uniform load over the entire truss there can be no stress in the braces; for we have only to remember what the equilibrium polygon signifies, to see that the bow will require no bracing to keep it in place. Or, since the stress in the lower member and the horizontal component of the stress in any piece of the bow are equal, their difference, the horizontal component of the stress in any brace, is zero: hence the stress itself is zero. Or, since the stress in the lower member is constant, there is no increment from any brace, and therefore no stress in any brace. It is right, consequently, that all of the verticals should be adapted to convey a tensile stress to the bow equal to whatever amount of load may be placed at their lower ends. It is also apparent that the steady load, being always uniformly distributed, will exert no stress on the braces, and may be neglected in their analysis, but must be subtracted from the compression which the rolling load may cause in the verticals. The assumption of uniform distribution of steady load is not strictly accurate; but the increase in the weight of the bow per *horizontal* foot, as we approach the abutments, is partially offset by the increase of weight per horizontal foot for the web members as we go towards the

middle. One may, therefore, include the steady load, or neglect it, in treating the braces, with the same result.

We have proceeded upon the assumption that the diagonals are ties, it being the usual and most economical construction to put the compression members on the shortest lines; but, in case the diagonals are struts, the verticals will always be in tension.

74. **Chord-Stresses when Verticals will not transmit Tension.** — In case the truss has no verticals, or where the verticals are not fastened at their ends so as to transmit tension (a mode of construction not to be commended), the weights at the loaded points must strain the diagonals on each side; and the stresses in the chords, while not varying much from the amounts previously deduced, will not have strictly a constant horizontal component. The simplest way to show the effect of the absence of verticals is to draw the stress diagram of Fig. 33 by the method explained in Part I, "Roofs," when the results are apparent at a glance. The weight of the bow may properly be left out of consideration, as it is practically in equilibrium by itself, and is exactly so as far as it is of uniform weight per horizontal foot. The horizontal thrust due to the weight of the bow and the equal tension in the lower chord must then be added to the results of this diagram. While, in any case, where there are three members at one joint, each capable of carrying the same kind of stress (as, in the bowstring girder of Fig. 32, we have two diagonals and a vertical meeting at each lower joint, and all designed to resist a possible tension), the paths which the concentrated weight at any joint may take in going to either abutment are somewhat doubtful, depending upon the cross-section, resistance to extension, and rigidity of attachment of the respective pieces, it is most natural that the weight should pass up the vertical to the bow; for in this line it meets the most direct resistance or reaction. That is, if we imagine the weight to strain all three of the pieces at once, a slight yielding, or extension of the two diagonals, will lower the loaded point more than the same amount of extension in the

vertical, and hence the weight will be thrown more upon the vertical; and as the vertical, being the shortest member, will stretch least in total amount for the same stress per square inch, the weight will still more be carried by it. Hence the assumption, that the verticals carry the load when it is complete, cannot be far from the truth.

75. **Maximum Stresses in Braces.** — By deducing a formula for the braces, we shall be enabled to prove a very short construction for obtaining the desired stresses. In finding these stresses, we will avail ourselves of the fact that the horizontal projection of the stress in a brace must equal the difference of the horizontal forces in the two chord-pieces of the panel in which the brace is situated; and we will prove that this horizontal component is a constant quantity when the brace experiences the maximum stress. If the steady load is neglected at present, the polygon A E F G I D B may represent the equilibrium polygon for a complete moving load. If the sides F G, G I, &c., are prolonged until they intersect the vertical through A, the points K, L, M, &c., will be the extremities of the equilibrium polygons for partial loads, extending from B to the successive joints. The distance A K has been proved, in § 68, to be $\frac{w' l}{H \cdot N}$ when w' = travelling load on one joint. As H will, by § 72, equal $\frac{N w' l}{8 k}$, the value of

$$A K = \frac{w' l}{N} \cdot \frac{8 k}{N w' l} = \frac{8 k}{N^2},$$

a quantity evidently independent of w', since it is obtained by construction from the bowstring girder itself.

If n represents the number of any panel and of any vertical from A, any distance A L or A M, intercepted on the vertical, will be seen, from § 68, to equal $\frac{n (n-1)}{2} \cdot \frac{8 k}{N^2}$. The height of any ordinate, P R, from the base line A B to the closing line B L of a particular polygon, will then be

$$\frac{n (n-1) 4 k}{N^2} \cdot \frac{(N-n)}{N}.$$

In the panel S G I R the stress in S R, if G R is the tie in action, will be $\frac{GQ}{GS}\cdot H$, and the horizontal force in G I will be $\frac{IP}{IR}\cdot H$. As the horizontal force in the piece of the bow exceeds that in the lower chord for the same panel, the desired difference of horizontal force will be

$$\left(\frac{IP}{IR}-\frac{GQ}{GS}\right)H=\left(\frac{IR-RP}{IR}-\frac{GS-SQ}{GS}\right)H=\left(\frac{SQ}{GS}-\frac{RP}{IR}\right)H.$$

As D C $= k$, any ordinate numbered n from A $=\frac{4k}{N^2}(N-n)\,n$: I R is the nth ordinate, and G S is the $(n-1)$st ordinate, when S R is the nth panel. The above expression becomes, by substitution,

$$\left[\frac{n\,(n-1)\,(N-n+1)}{(N-n+1)\,(n-1)}-\frac{n\,(n-1)\,(N-n)}{(N-n)\,n}\right]\frac{4k}{N^3}\cdot\frac{N^2}{4k}\cdot H=\frac{H}{N}=\frac{w'l}{8k},$$

which is a constant quantity.

If, then, we construct a parabola divided into N panels, each $\frac{w'l}{8k}$ long, the entire span being H for rolling load, and the height proportional to the height of the original truss, or if, in other words, we draw the truss anew to this altered scale, the diagonal lines will be parallel to the diagonals of the truss, and, having the required constant horizontal projection, will be the desired tensions in those diagonals. The required middle ordinate of the parabola will be obtained by the proportion

$$l:k=\frac{N\,w'\,l}{8\,k}:\frac{N\,w'}{8};$$

or it is one-eighth of the entire travelling load. All of these values may be obtained, but with more labor, by the general method.

This truss is drawn below the original truss in Fig. 32, and is lettered in small type to indicate the stresses in the respective diagonals. It will be seen that diagonals sloping each way are required in all panels; that the diagonals in the middle of the span have the most stress; and that all diagonals which have the same vertical height have the same maximum stress.

76. **Stresses in the Verticals.** — An inspection of the diagrams of Fig. 30 shows, that, since the bottom chord of the bowstring girder is horizontal, the vertical which transfers the stress from the diagonal to the first lightly-loaded joint carries all of the shear from that panel except what is taken by the upper chord in the next lightly-loaded panel. If, then, we find the horizontal component of the stress in that piece of the chord and the shear, we can draw our diagrams. In the first place, it is apparent that the horizontal component of the stress in F G is the same with that in S R. From the last section, the horizontal component in S R $= \frac{G\,Q}{G\,S}\,H = \left(1 - \frac{S\,Q}{G\,S}\right) H$. By inspection of the value for the ratio $\frac{S\,Q}{G\,S}$ written above, and cancelling factors, we get

$$\text{Horizontal component in F G} = \left(1 - \frac{n}{N}\right) H.$$

If we draw a horizontal line from F to meet G S, we see that

$$\text{Vert. comp. in F G} : \text{hor. comp. in F G} = G\,S - F\,T : \frac{l}{N}.$$

G S being the $(n-1)$st ordinate, and F T the $(n-2)$d ordinate,

$$G\,S - F\,T = \frac{4\,k}{N^2}\left[(N - n + 1)\,(n - 1) - (N - n + 2)\,(n - 2)\right]$$
$$= \frac{4\,k}{N^2}\,(N - 2\,n + 3):$$

therefore

$$\text{Vert. comp. in F G} = \left(1 - \frac{n}{N}\right)\frac{N\,w'\,l}{8\,k} \cdot \frac{4\,k}{N^2}\,(N - 2\,n + 3)\,\frac{N}{l}$$
$$= \frac{(N - n)\,(N - 2\,n + 3)}{2\,N}\,w'.$$

As the travelling load alone is now under consideration, the shear on the left of the loaded portion will be constant, and equal to the reaction at A, which is

$$\frac{(N - n)\,w'\,(N - n + 1)}{2\,N}.$$

The amount of compression in the vertical G S will be the shear in the panel S R minus the vertical component in F G, or

$$\frac{N - n}{2\,N}\,w'\,(N - n + 1 - N + 2\,n - 3) = \frac{(N - n)\,(n - 2)}{2\,N}\,w'.$$

As G S, by our notation, is the $(n - 1)$st vertical, the compression in the nth vertical will be obtained by writing $(n + 1)$ for n, or

$$\text{Compression in } n\text{th vertical} = \frac{(N - n - 1)\,(n - 1)}{2\,N}\,w'$$
$$= \frac{(N - n)\,n}{2\,N}\,w' - \frac{N - 1}{2\,N}\,w'.$$

The height or rise of our parabola for stresses in diagonals was found to be $\frac{N\,w'}{8}$. The nth ordinate of this parabola will be $\frac{N\,w'}{8}\cdot\frac{4}{N^2}\,(N - n)\,n = \frac{(N - n)\,n}{2\,N}\,w'$; and the first ordinate, when $n = 1$, is $\frac{N - 1}{2\,N}\,w'$. We see, therefore, that, if a horizontal line is drawn through e at the height of the first ordinate above the base, the remaining portion of each vertical above this line will be the compression from travelling load upon the corresponding vertical of the girder. Another horizontal line, $u'\,k'$, at an altitude above the one just drawn of the steady load w at the foot of each post, neglecting the weight of the bow, as before stated, will cut off the stress of tension due to the steady load, and the remainder will be the maximum compression in each vertical. The result is, that the first vertical is never compressed, but has a tension of $w + w'$; the second vertical will not be compressed unless w is less than the difference between the first two verticals of the stress parabola; and the remainder are compressed, except, again, the last two.

77. **Recapitulation.** — These constructions are all found in Fig. 32, and may be briefly summed up as follows: Having constructed the skeleton A D B of the truss, make D J = C D, and draw A J, the tangent to the parabola at the abutment. Lay off 2–3 vertically, equal to one-half of the weight of truss when

fully loaded, and divide it, at 4, 5, . . . 8, into panel weights beginning and ending with a half-load. Draw 2–0, parallel to J A, to meet the horizontal line 3–0: 4–0, 5–0, &c., are the stresses in the pieces A E, E F, &c., of the bow; and the stress in the bottom chord is 3–0. The value of 3–0 can be checked by the formula $\frac{W'' l}{8 k}$. Draw the truss anew to a different scale, the panel length being $\frac{w' l}{8 k}$, or the span $= \frac{N w' l}{8 k}$, and the rise $c d = \frac{N w'}{8}$. The diagonals will be the tensions in the corresponding diagonals of the truss. Draw $u' k'$ horizontally at a distance $e u' =$ one panel weight of steady load minus one panel weight of bow: the ordinates from this line to the bow will be the stresses in the verticals, — compression when above the horizontal line, tension when below. If $k' l'$ is maximum load which can come on each lower joint, it equals possible tension in each vertical.

78. **Triangular Bracing.** — It is not uncommon to introduce the arrangement of bracing shown in Fig. 34 into the bowstring truss. The chord-stresses may then be most readily obtained by the method employed in Fig. 33, when, as there is but one system of braces, the small parallelograms $a\ b\ d\ c$, &c., due to the intersections A B D C, &c., will disappear. An adaptation of the construction for diagonal stresses in Fig. 32 will probably give the alternating compressions and tensions in the diagonals, so far as they are due to travelling load; and to these stresses may be added *algebraically* that fraction of the diagonal stresses previously obtained in the chord-stress diagram, which is properly due to steady load. It is hardly expedient to take up space with a detailed analysis: any reader who is specially interested in such a type of truss can, from the general methods already laid down, elaborate a set of diagrams for himself. If the bow is spaced off into pieces of equal length, the stresses on the various parts will be modified by the change. As the method of Fig. 30 applies to any of these trusses, it will

doubtless be satisfactory when required for occasional use. Any special method which is not so simple as to be readily recalled, but which must be reviewed whenever it is needed for use, will only be valuable to those, who, from frequent necessity, can keep it freshly in mind.

79. **Bowstring Girder with Circular Bow.** — The same general remarks apply to the girder with a bow in a circular arc. As the ordinates to a circular arc from the chord of the arc are not expressed in any simple terms of the span and rise of the truss, and as these latter quantities have no fixed ratio, it is doubtful whether any short construction can be devised for finding the web-stresses. The general method is perfectly applicable. Special care is necessary in drawing the lines parallel to the pieces of the chord in Fig. 31, because they are many of them long, and a slight inaccuracy would affect the ascertained magnitude of the stresses in the braces. Good results can be obtained with the girder of the present section, since the effective lines of all the pieces of the bow are perpendicular to radii drawn to their middle points, and hence their directions can be accurately ascertained. A similar precaution applied to the parabola, of drawing tangents or long chords, will give the direction of the pieces of the bow closely. When the girder with a circular bow has a comparatively large rise, the web stresses do not, as in the parabola, increase from the abutments to the middle. As the bow will not be in equilibrium under a load distributed uniformly horizontally, the steady load will affect the braces, and must be taken into account in that connection.

80. **General Remarks.** — In some cases these trusses are inverted, as in Fig. 35, making the suspended bowstring truss. And, where the locality permits, that form of truss offers some theoretical advantage: the compression members will occupy the shortest lines, the bow will resemble a suspension-bridge cable, and the truss, being in stable equilibrium, will require but moderate lateral bracing in a vertical plane. Occasionally a designer is tempted to draw normals to the curve of the bow

in place of verticals from the joints of the horizontal member: such an arrangement is of doubtful utility in the parabolic girder, which is in equilibrium under vertical forces distributed uniformly per horizontal foot. The circle being a curve of equilibrium under normal forces, of uniform intensity along the arc, such a construction may be more defensible for a circular bow. It may also be advantageous, for constructive reasons, to have all the pieces of the circular bow of one length. Verticals will then divide the chord into unequal parts, shortest near the abutments: normals will also divide the chord into unequal lengths, shortest at mid-span. The change thus produced in the magnitude of the load at each joint must be regarded. In a few instances both chords have been parabolas or circular arcs, with the convexity turned opposite ways. The analysis would present no difficulty.

All iron trusses should have rollers, or other means of movement, at one abutment, to allow for elongation and contraction under changes of temperature.

81. **Extent of Continuous Load to produce Maximum Stress in a Diagonal of any Truss.** — In determining the maximum stress in any brace of a truss, we have supposed that the rolling load extended from one abutment to the panel in question, and it was proved that such a disposition of load gave the greatest shear in the panel. It is manifestly impossible, however, to so dispose a *continuous* load that one panel joint shall sustain a full panel weight of moving load, while the next joint in advance carries none; for the load which comes on a joint must be thrown upon it by the floor-timbers or track-stringers on one or both sides; and, as soon as the head of the load advances beyond any joint, the next joint beyond begins to carry something: therefore no joint can be fully loaded until the load has covered the panel beyond. The shear in that panel, then, can never be quite so large as the usual assumption would make it: the results previously obtained will err a little on the side of safety, and the treatment is defensible on that account. Loads imposed by railroad-trains are not con-

tinuous, but are concentrated on the wheels. How far apart these may be will depend upon the train: in one extreme we have coal-cars, and in the other palace-cars. In some cases the loaded points may be a panel length or less apart; in many cases they are distant more than a panel length. A special treatment may be devised with such a distribution of the load as is deemed best for any particular problem; but the customary assumption is that of loads concentrated at panel joints, and the first loaded joint completely loaded, as we have treated it.

As the actual shear will be slightly less than the value found by the above assumption, by reason of the subtraction of the fractional weight transferred to the joint in advance, it may be interesting to offer an analytical investigation into the extent of a uniform load which shall produce the maximum stress on a brace of a truss of any form. The general steps of this treatment, substantially as here given, were contributed by S. W. Salmon to "The Analyst" for February, 1874.

Let the span A B of the truss, Fig. 36, $= l$, the distance A C to any joint $= a$, and the distance A D to the next joint $= b$; let the height C F of the truss at C $= h'$, and of D E at D $= h''$. Let H' = horizontal force at the section at C, and H'' = the same at D. The difference between H' and H'' is the horizontal component of the stress in D F. Let w' = moving load per unit of length extending from A a distance x. The upward reaction at B $= \frac{w'\,x^2}{2\,l}$. We may have three expressions for H' — H''; one when x is less than a, one when x is greater than b, and the last when the value of x lies between a and b. In the first two cases H' — H'' will increase as x approaches a and b respectively, and there is no absolute maximum. But the last case has a value of x which makes H' — H'' a maximum. At that time

$$H' = \frac{w'\,x^2}{2\,l} \cdot \frac{(l-b)}{h''};\ H' = \left\{\frac{w'\,x^2}{2\,l} \cdot (l-a) - \frac{w'\,(x-a)^2}{2}\right\} \div h'$$

$$= \frac{w'}{h'}\left(a\,x - \frac{a\,x^2}{2\,l} - \frac{a^2}{2}\right);$$

$$H' - H'' = \frac{w'}{h'}\left(a\,x - \frac{a\,x^2}{2\,l} - \frac{a^2}{2}\right) - \frac{w'}{h''}\,\frac{(l-b)\,x^2}{2\,l},\ \text{to be made a maximum.}$$

Differentiating relatively to x, and putting the first differential co-efficient equal to zero, we get

$$\frac{a}{h'} - \frac{a\,x}{h'\,l} - \frac{(l-b)\,x}{h''\,l} = 0,$$

$$x = \frac{a\,l\,h''}{a\,h'' + (l-b)\,h'}.$$

Substitute in the value of $H' - H''$, which then becomes

$$H' - H'' \text{ (max.)} = \frac{w' a^2}{2} \left\{ \frac{(l-a)\, h'' - (l-b)\, h'}{a\, h'\, h'' + (l-b)\, h'^2} \right\}, \text{ in the most general form.}$$

82. **Applications.** — If the truss has parallel chords, $h' = h'' = h$, and

$$H' - H'' \text{ (max.)} = \frac{w' a^2}{2\, h} \left\{ \frac{b-a}{l-(b-a)} \right\}.$$

If $b - a = p$, a constant panel length or half-panel length, as the case may be, $l = \text{N}\, p$, and $a = n\, p$:

$$H' - H'' = \frac{w' a^2 p}{2\, h\, (l-p)} = \frac{w' n^2 p^2}{2\, h\, (\text{N}-1)}.$$

The shear F, which is the other component of the stress in the brace, will be to $H' - H''$ in the ratio $\frac{h}{p}$: hence

$$\text{F (max.)} = \frac{w' n^2 p}{2\, (\text{N}-1)}.$$

This expression is the ordinate to a parabola which is convex to the horizontal axis, has its vertex at one abutment and its extreme ordinate, when $a = n\, p = (\text{N}-1)\, p$, becomes $\frac{\text{N}-1}{2}\, w'\, p$, coincident with the ordinate for travelling load in the middle of the first panel which we have heretofore used. To ascertain how much of the truss must be successively loaded to cause the maximum shear in each panel, change the value of x by dividing out the common factor $h' = h''$ and making substitutions,

$$x = \frac{a\, l}{l-(b-a)} = \frac{a\, l}{l-p} = \frac{n\, p\, l}{\text{N}\, p - p} = n\, \frac{l}{\text{N}-1}.$$

If, therefore, the span is divided into $\text{N}-1$ equal parts, the maximum shear in any panel will occur when the load extends up to the division in that panel, as shown in Fig. 37, and it will equal the ordinate to the *parabola* at that point. To the above expression for F must be added the shear for steady load. It is to be measured in the middle of the panel, as heretofore; and the two ordinates must, therefore, be measured separately.

If the truss is parabolic, the height h, at any point, will be, if k equals the rise at the middle,

$$h = \frac{4\, k}{l^2}\, x'\, (l - x');$$

and we obtain by substitution, making $x' = a$ or b, as required,

$$H' - H'' \text{(max.)} = \frac{w' a^2}{2} \cdot \frac{l^2}{4\, k} \left\{ \frac{(l-a)\, b\, (l-b) - (l-b)\, a\, (l-a)}{a^2\, b\, (l-b)\, (l-a) + a^2\, (l-b)\, (l-a)^2} \right\}$$

$$= \frac{w' l^2}{8\, k} \left\{ \frac{b-a}{b+l-a} \right\} = \frac{w' l^2}{8\, k} \cdot \frac{p}{l+p} = \frac{w' l^2}{8\, k} \cdot \frac{1}{\text{N}+1},$$

which is a constant quantity, as in the earlier investigation of the bowstring girder, w' in that case denoting a panel weight.

If the value of x is again reduced for this truss to obtain the extent of load to cause the maximum shear in each panel, it becomes

$$x = \frac{a\,l\,b\,(l-b)}{a\,b\,(l-b) + (l-b)\,a\,(l-a)} = \frac{l\,b}{b+l-a} = \frac{b\,l}{l+p} = (n+1)\,\frac{l}{N+1},$$

where the distance b exceeds a one panel length. If, then, the span of Fig. 38 is divided into $N+1$ equal parts, the load, when extending up to the successive points of division, will give the maximum stress in the diagonal which crosses the head of the load. A diagram can therefore be constructed, as before, of the constant panel length $\frac{w'\,l^2}{8\,(N+1)\,k}$, by which to obtain the diagonal stresses.

CHAPTER VI.

FLEXURE AND DEFLECTION OF BEAMS.

83. **Flexure of Beams, &c.** — The bending moment at any section of a beam is opposed or balanced by the moment of resistance of the fibres at the section. Under the tensions and compressions to which the fibres are exposed, the particles of the beam are extended in one portion of the section, and compressed in the other; so that a curvature of the beam results, with a change in direction, in a vertical plane at that point of the centre line or axis of the beam, as represented and exaggerated in Fig. 39: therefore the change of direction or inclination at any point is proportional to the bending moment at that point, or to the product of H by the ordinate between the equilibrium polygon and the straight line. By reference to Fig. 39 we can see the change of inclination produced in the beam at A by the elongation of the upper fibres at that point, and the compression of the lower ones. We can also see the effect of successive changes of inclination at B, C, and D, in altering the direction of the remainder of the beam; and we note that the changes of inclination at E, F, &c., produced by bending moments in the opposite direction, tend to bring the beam back toward its original *direction.*

84. **Change of Inclination.** — The total change of inclination between any two points is proportional, therefore, to the *sum* of all the bending moments between those points, or to the *area* included between the equilibrium polygon, the closing line, and the two limiting ordinates under the points. If portions of the bending moments are of opposite kinds, the *algebraic* sum of

the ordinates is to be taken, or the difference of areas of contrary signs. As the angle of inclination is measured by the ratio of the vertical movement of the end of the beam to the horizontal distance from the apex of the angle, the numerical quantity to be obtained will be the tangent of the angle which the tangent to the bent beam at any point makes with its original horizontal position.

85. **Modulus of Elasticity.** — But it is manifest that the amount of flexure of a beam will be influenced by the material of which it is made. Let us imagine two bars — one of iron, the other of wood — of the same length and the same cross-section, firmly held horizontally by one end, and having equal weights attached at the free ends. The movement of the two bars below the original horizontal line, for similar points, will not be the same, nor will the changes of inclination. The bar which elongates and compresses most for a given stress on the square inch of section will bend the most as a beam under a given load; and therefore the *ratio of the force on a square inch of section to the elongation or compression produced in a piece an inch long* influences the flexure of any piece under a bending moment. This ratio, known technically as the *modulus of elasticity*, and denoted by the symbol **E**, is deduced from experiments upon flexure, or upon extension under a tensile stress, and varies with the material. As the change of length is the denominator of this ratio, the more rigid body, the one which bends less for a given bending moment, other things being equal, will have a higher modulus of elasticity: therefore the curvature, the change of inclination, and deflection of each point from its original position, will vary *inversely* as this modulus; its values will be found in tables of strength of materials.

86. **Moment of Inertia.** — Again: if we compare the action of two beams of the same length and material, of different but similar cross-sections, we know that they may not be equally affected by equal weights similarly placed; that is, by equal bending moments. If they are of equal depths, the broader

y p = stress at unit distance

stress at $y = p\,y$

Q $dA = dy \cdot dz$

Z stress on $Q = p \cdot y \cdot dy \cdot dz$

n. a

$dM_z = p \cdot y^2 \cdot dy \cdot dz$ $\therefore M = p\,I$

$dA = b \cdot dy$

$dI = y^2 b\, dy$

$I = b \sum y^2 dy = b \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 dy$ $\quad \mathbf{I} = \frac{b h^3}{12}$

$= b \left[\frac{y^3}{3} \right]_{-\frac{h}{2}}^{\frac{h}{2}}$

one will be the stiffer; and, if the breadths are the same, the deeper beam will be much the stiffer. Referring to Fig. 24 of Part I., "Roofs," with the explanation of that figure there given, we see that the moment of resistance is made up of the summation of the products of the stress on each particle, multiplied by its distance from the centre line, or neutral axis, where the stress changes from tension to compression. As the stress on each particle increases with the distance from the centre line, the same moment of resistance might be represented by the stress on a particle at a unit's distance multiplied by the summation of the products of the area of each particle by the *square* of its distance from the centre line. As the stress upon the extreme or most remote particle in a section will be less for a given moment of resistance the deeper the beam is, the flexure of the beam will vary inversely as the summation last spoken of. This summation of the area of each particle, multiplied by the square of its distance from the centre line, is known as the *moment of inertia* of the section, and is denoted by **I**.

It may be obtained from the moment of resistance, deduced graphically in Part I., "Roofs," p. 57, by dividing by the stress on a square inch at a unit's distance, which is equal to the stress, f, on the extreme fibre per square inch, divided by the distance from that fibre to the neutral axis, usually one-half the depth of the beam. For example: The moment of resistance of a rectangular cross-section was shown to be $\frac{1}{6} f b h^2$, where $b =$ breadth, and $h =$ height of the section: the moment of inertia of a rectangle will, therefore, be

$$\mathbf{I} = \tfrac{1}{6} f b h^2 \div \frac{2f}{h} = \tfrac{1}{12} b h^3.$$

87. **Formula for Change of Inclination.** — The *change* of inclination at any one point of a beam will, therefore, be

$$\frac{\mathrm{M}}{\mathrm{E\,I}} = \frac{\mathrm{H}\, y}{\mathrm{E\,I}},$$

where y equals the ordinate to the equilibrium polygon at the point; and the total change of inclination between any two

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$I = \int_0^{\frac{h}{2}} y^2 x\, dy$

points will be, if i equal angle of inclination to the horizon, and Σ is the sign of summation,

$$tan.\ i = \Sigma\, \frac{H\, y}{E\, I}.$$

Of the three quantities involved, M, the bending moment, depends upon the load and its distribution, thus including the span of the beam: **E**, the modulus of elasticity, depends upon the material; and **I**, the moment of inertia, upon the form of cross-section. All the variables are thus included in the general expression.

88. **Deflection; Area Moments.** — Upon reference to Fig. 40 it is also evident that the vertical *deflection* of any point of a beam from the original horizontal line depends upon the several changes of inclination, and the distances of the points at which they occur from the above point. Thus, if an originally straight rod $a\,g$ is bent at a, the point g will be carried to a point on the line $a\,l$; the distance through which it is displaced depending upon the angle at a and the distance $a g$. If another angle is made at b, the point g will be found on $b\,i$; and, on a further bending at c, it will move to the direction $c\,h$. The changes of inclination at d, e, and f, in the contrary direction, will carry the point which was originally at g through $d\,k$, $e\,m$, $f\,g$, finally back again to g. (The deflections of all beams and trusses are so small, that the curved line of a beam under a load is always considered practically equal in length to the horizontal distance between the two points of support.)

As the expression of the last section measures the angle at a, it will only be necessary to multiply it by the horizontal distance of g from a to find the vertical displacement of g by reason of the change of inclination at a. The modulus of elasticity and the moment of inertia will affect the deflection as they do the changes of inclination. It is then evident that the total deflection or displacement vertically of any point, such as d, from the straight line tangent to the curve of the beam at a, will be obtained by summing, from a to d, the products of the

bending moment at each point of the beam multiplied by the horizontal distance of each point from d, and dividing the sum by **E I**. If x denotes the distance of any point horizontally from d, the above deflection, v, may be written

$$v = \Sigma_d^a \frac{\mathrm{M}\, x}{\mathbf{E\,I}} = \Sigma_d^a \frac{\mathrm{H}\, y\, x}{\mathbf{E\,I}},$$

where the letters attached to the sign of summation Σ denote that the addition of products is to extend from a to d.

In the same way that the summation of the ordinates y to the equilibrium polygon gives an area, the summation of the products of each ordinate into its distance from d is equal, on the principle that the moment of a resultant equals the sum of the moments of the components, to the product of the area just referred to by the *distance of its centre of gravity horizontally* from d. To this last product we give the name of *area moment.* As a convenient aid in remembering in which direction the horizontal distances are to be measured, we may note, that, if we regard the angle made by the original line of the beam and its new direction, the measurement is to be made away from the vertex, towards the opening of the angle.

89. **Mathematical Solution.** — Another demonstration of the above theorems, which is brief, and which depends upon the usual expressions for the investigation of curvature, slope, and deflection of beams, by mathematical analysis, is as follows: —

Let M denote the bending moment at any point of a beam supported in any way. Let **E** denote the modulus of elasticity of the material; and **I**, the moment of inertia of the cross-section. Let the originally straight horizontal line of the beam be the axis of x, and let v be measured vertically. M will be a function of x. Let r equal radius of curvature of the axis or centre line of the bent beam at any point. Then we may write the well-known equation for the curvature,

$$\frac{1}{r} = \frac{d^2 v}{d x^2} = \frac{\mathrm{M}}{\mathbf{E\,I}}.$$

If we integrate this expression once, considering **I** constant, we have

$$\frac{d v}{d x} = \frac{1}{\mathbf{E\,I}} \int \mathrm{M}\, d x = \text{tan. inclination.}$$

If we determine and introduce the constant of integration, we find the incli-

nation of the beam at each point to the horizon; but if we integrate from 0 to x, the origin being taken at one of the points of support, we get a complete integral, — the area included between the equilibrium polygon and the closing line, — but one expressing only the change of inclination from the slope already existing at the origin, or the angle between the tangent at any point and the tangent at the origin.

Integrating again, we have,

$$v = \frac{1}{\mathrm{E\,I}} \int\!\!\int \mathrm{M}\, d\, x^2 = \text{deflection}.$$

This integral is a volume, and taken between limits, as before, is the summation of each area from 0 to x into a height $d\,x$, giving a *cone* with a base equal to $\int_0^x \mathrm{M}\, d\,x$ and a height x: it is also equal to the area $\int \mathrm{M}\, d\,x$ multiplied by the distance of its centre of gravity from the point whose abscissa is x. **I** is here considered constant, and may be so taken in most trusses. If **I** varies, it will be expressed in terms of x, and retained within the integral sign.

90. **Applications.** — The areas are readily measured by scaling equidistant ordinates, and multiplying by the constant distance between two ordinates, as is done in calculating the contents of any irregular area by offsets. For a continuous load the equilibrium polygon becomes a curve, and the included areas with their centres of gravity are easily obtained for all practical cases.

We will now proceed to find the inclination, or slope, and the deflection, of several beams loaded in different ways. As **E** and **I** are usually expressed in units of pounds and inches, these units must be employed in denoting the imposed weights, the weight and dimensions of the beam. In all cases, l will denote the length or span of the beam; W, a single concentrated load; and w, the intensity per inch of a distributed load. The reader will see, after an inspection of one or two cases, what quantities are successively multiplied together.

91. **1st, Beam fixed at One End, loaded at the Other.** — The beam built into a wall or otherwise fixed at one end, and carrying a weight W at the free end, will take the form of the dotted curve sketched in Fig. 41. If W is laid off on a vertical line, it will represent the load, and also the shear at any

$i = \frac{Wa^2}{2EI}$ $EI\frac{d^2y}{dx^2} = W(a-x)$ $EIy = W\left(\frac{ax^2}{2} - \frac{x^3}{6}\right) + C$

$EI\frac{dy}{dx} = W\left(ax - \frac{x^2}{2}\right) + C$ $EI\,d = W\left(\frac{a^3}{2} - \frac{a^3}{6}\right)$

Integration each line

point of the beam from this load. Drawing H from one end, and completing the stress diagram, we see that the equilibrium polygon is a right-angled triangle, the bending moment increasing simply as the distance from the free end of the beam. Let k denote the height of this triangle. Then, by proportion, we shall have

$$W : H = k : l; \text{ or } Hk = Wl = M \text{ (max.)}.$$

The area of the triangle $= k \cdot \frac{1}{2} l$. If the beam is of uniform cross-section, **I** is constant. We have, then, for the slope at the extreme end, where the weight is attached,

$$tan.\ i = \Sigma_o^l \frac{H}{EI} y = \frac{H}{EI} \Sigma_o^l y = \frac{1}{EI} Hk.\ \tfrac{1}{2} l = \frac{Wl^2}{2EI}.$$

The centre of gravity of the triangle being distant $\frac{2}{3} l$ from the apex horizontally, the deflection will be obtained by multiplying *tan.* i by this distance, or

$$v = \frac{Wl^2}{2EI} \cdot \tfrac{2}{3} l = \frac{Wl^3}{3EI},$$

which is the vertical distance that the point of attachment of the weight is below the tangent at the fixed end.

92. **2d, Beam fixed at One End, and uniformly loaded.**—If the uniform load over the whole extent of the beam is considered as concentrated at a series of equidistant points, the equilibrium polygon will be readily drawn; and the true curve, when the polygon's sides are increased in number indefinitely, will be seen to be a parabola, Fig. 42, with its vertex below the free end of the beam, and tangent to the closing line. As this curve is in equilibrium, the two tangents at the extremities must, by § 10, meet on the vertical through the centre of gravity of the load; that is, at $\frac{1}{2} l$ from one end. Therefore, upon drawing the tangent at the left extremity of the curve, which will be parallel to the most inclined line of the stress diagram, we get, by similar triangles,

$$wl : H = k : \tfrac{1}{2} l; \text{ or } Hk = \tfrac{1}{2} wl^2 = M \text{ (max.)}.$$

As the area of a parabolic segment is two-thirds of the enclosing rectangle, the area of this figure will be $\frac{1}{3} l$ multiplied

by its altitude k; its centre of gravity lies at $\frac{3}{4}\,l$ from the right: hence, as before,

$$tan.\ i = \frac{1}{\mathrm{E\,I}} \cdot \tfrac{1}{2}\, w\, l^2 \cdot \tfrac{1}{3}\, l = \frac{w\, l^3}{6\,\mathrm{E\,I}};$$

$$v = \frac{w\, l^3}{6\,\mathrm{E\,I}} \cdot \tfrac{3}{4}\, l = \frac{w\, l^4}{8\,\mathrm{E\,I}}.$$

93. **3d, Beam fixed at One End, both uniformly loaded and loaded at Free End.** — If the beam is supposed to carry a weight W at the free end in addition to its own distributed weight, we may combine the two preceding figures, and therefore add the above expressions for slope in the one case, and for deflection in the other. Thus we obtain

$$tan.\ i = \frac{1}{\mathrm{E\,I}} \left(\frac{w\, l^3}{6} + \frac{\mathrm{W}\, l^2}{2} \right);$$

$$v = \frac{1}{\mathrm{E\,I}} \left(\frac{w\, l^4}{8} + \frac{\mathrm{W}\, l^3}{3} \right).$$

94. **4th, Beam supported at Both Ends, a Single Weight in the Middle.** — The equilibrium polygon is drawn in Fig. 43, and the reaction at each abutment is $\frac{1}{2}$ W. On account of symmetry of loading, the beam will be horizontal at the middle, the greatest slope will be found at either abutment, and the deflection at the middle will be equal to the elevation of the end of the beam above the horizontal tangent at the middle: hence

$$\tfrac{1}{2}\,\mathrm{W} : \mathrm{H} = k : \tfrac{1}{2}\, l;\ \text{or}\ \mathrm{H}\, k = \tfrac{1}{4}\,\mathrm{W}\, l = \mathrm{M}\ (\text{max.}).$$

Reckoning the change of inclination from the middle to one end, we have for the area a triangle of area $k \cdot \frac{1}{4}\, l$, and

$$tan.\ i = \frac{1}{\mathrm{E\,I}} \cdot \tfrac{1}{4}\,\mathrm{W}\, l \cdot \tfrac{1}{4}\, l = \frac{\mathrm{W}\, l^2}{16\,\mathrm{E\,I}}.$$

Remembering that the horizontal distance from the centre of gravity is to be measured towards the opening of the angle between the tangent and the beam, we find

$$v = \frac{\mathrm{W}\, l^2}{16\,\mathrm{E\,I}} \cdot \tfrac{1}{3}\, l = \frac{\mathrm{W}\, l^3}{48\,\mathrm{E\,I}}.$$

95. **5th, Beam supported at Both Ends, and uniformly loaded with $w\, l$.** — The equilibrium curve, Fig. 44, will be the

well-known parabola. The tangent at one end, as explained in § 72, will cut the middle ordinate prolonged at $2\,k$ from the horizontal line: hence

$$\tfrac{1}{2}\,w\,l : H = 2\,k : \tfrac{1}{2}\,l; \text{ or } H\,k = \tfrac{1}{8}\,w\,l^2 = M \text{ (max.)},$$

as has previously been shown. Then from the parabolic area, as above explained, we get

$$tan.\ i = \frac{1}{E\,I} \cdot \tfrac{1}{8}\,w\,l^2 \cdot \tfrac{2}{3} \cdot \tfrac{1}{2}\,l = \frac{w\,l^3}{24\,E\,I}.$$

The centre of gravity of the semi-segment of the parabola is $\frac{5}{8}$ of $\frac{1}{2}\,l$ from the abutment: hence

$$v = \frac{w\,l^3}{24\,E\,I} \cdot \tfrac{5}{16}\,l = \tfrac{5}{384}\ \frac{w\,l^4}{E\,I}.$$

It is worthy of notice, that with beams of the same length and the same *total* applied load, under Cases 1, 2, 4, and 5, the maximum bending moments are relatively as 1, $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{8}$; and the relative strengths are as the reciprocals, 1, 2, 4, and 8.

96. **6th, Beam supported at Both Ends, carrying a Single Weight, distant a from the Right.** — This case is represented by Fig. 45, and is given as a sample of the flexibility of the method. a is greater than $\frac{1}{2}\,l$. The reaction at the left point of support will be $\frac{W\,a}{l}$. Then by proportion, as usual,

$$\frac{W\,a}{l} : H = k : l - a; \text{ or } H\,k = \frac{W\,a\,(l - a)}{l} = M \text{ (max.)}.$$

The point where the beam is horizontal is at present unknown; but at that point, which will not be at the weight, is manifestly the maximum deflection. Suppose that the point is C, distant x from B. The distance which C is below A will be equal to $\frac{H}{E\,I}$ multiplied by the area moment of the area in the equilibrium polygon to the *left* of the dotted line below C. The area moment to the *right* of the dotted line multiplied by the same quantity will be the deflection of C below B. As the tangent at C is horizontal, these two expressions must be equal: hence to find the point of maximum deflection resolves itself into so dividing the equilibrium polygon by a vertical line, that the area

moment on one side, about the abutment on that side, shall equal the similar moment on the other side about its abutment. The quantity $\frac{\mathrm{H}}{\mathrm{E\,I}}$, being constant, will not affect x.

The dotted line will cut off a trapezoid from the triangle to the right of the weight. One of its parallel sides being k, the other will be given by the proportion

$$a : k = x : \frac{k\,x}{a},$$

and its area will be equal to one-half the sum of its two parallel sides multiplied by $a - x$, the perpendicular distance between them: hence the area of the trapezoid is

$$\tfrac{1}{2}\left(k + \frac{k\,x}{a}\right)(a - x) = \tfrac{1}{2}\,k\,\frac{a^2 - x^2}{a}.$$

By taking moments about F for the triangle F D I, we see that D I F . $\frac{2}{3}\,a$ — E G F . $\frac{2}{3}\,x$ = D I G E multiplied by x', the distance of centre of gravity from F, or, in symbols,

$$\frac{k\,a}{2} \cdot \tfrac{2}{3}\,a - \frac{x}{2} \cdot \frac{k\,x}{a} \cdot \tfrac{2}{3}\,x = \tfrac{1}{2}\,k \cdot \frac{a^2 - x^2}{a} \cdot x';$$

$$\tfrac{1}{3}\,k\,\frac{a^3 - x^3}{a} = \tfrac{1}{2}\,k\,\frac{a^2 - x^2}{a} \cdot x', \text{ or}$$

$$x' = \tfrac{2}{3}\,\frac{a^3 - x^3}{a^2 - x^2};$$

and the distance of the centre of gravity of the trapezoid from $\mathrm{A} = l - x'$: hence, making the area moment of the small triangle plus the trapezoid about A equal to the moment of the remaining area about B, we have

$$\tfrac{1}{2}\,k\,(l - a) \cdot \tfrac{2}{3}\,(l - a) + \tfrac{1}{2}\,k\,\frac{a^2 - x^2}{a}\left(l - \tfrac{2}{3}\,\frac{a^3 - x^3}{a^2 - x^2}\right) = \tfrac{1}{2}\,\frac{k\,x^2}{a} \cdot \tfrac{2}{3}\,x. \quad \text{(b)}$$

Dropping common factors, we get

$$(l - a)^2 + \frac{a^2 - x^2}{a}\left(\tfrac{3}{2}\,l - \frac{a^3 - x^3}{a^2 - x^2}\right) = \frac{x^3}{a}$$

$$a\,(l - a)^2 + \tfrac{3}{2}\,l\,(a^2 - x^2) - a^3 + x^3 = x^3$$

$$\tfrac{3}{2}\,l\,x^2 = a\,l^2 - \tfrac{1}{2}\,a^2\,l$$

$$x = \sqrt{\tfrac{1}{3}\,a\,(2\,l - a)}.$$

Substituting this value of x in the second member of the deflection equation (b) from which it was deduced, we see that

$$v = \frac{\mathrm{H}}{\mathbf{E\,I}} \cdot \tfrac{1}{3} \frac{k\,x^3}{a} = \frac{\mathrm{W}\,a\,(l-a)}{l\,k\,\mathbf{E\,I}} \cdot \tfrac{1}{3} \frac{k}{a} \cdot \tfrac{1}{3}\,a\,(2\,l-a) \sqrt{\tfrac{1}{3}\,a\,(2\,l-a)}$$

$$= \frac{\mathrm{W}\,(l-a)\,a\,(2\,l-a)}{9\,\mathbf{E\,I}\,l} \sqrt{\tfrac{1}{3}\,a\,(2\,l-a)};$$

which expression, when $a = \frac{1}{2}\,l$, reduces to $\dfrac{\mathrm{W}\,l^3}{48\,\mathbf{E\,I}}$, as in Case 4.

The slope at B will be

$$tan.\ i = \frac{\mathrm{H}}{\mathbf{E\,I}} \cdot \frac{k\,x}{a} \cdot \tfrac{1}{2}\,x = \frac{\mathrm{W}\,(l-a)\,a}{6\,\mathbf{E\,I}\,l}\,(2\,l-a).$$

The slope at A will be obtained similarly.

97. **Beam on Two Supports, but overhanging.**—Before taking up additional cases of deflection of beams, it may be well to discuss the equilibrium polygon for a beam loaded in any manner, carried on two supports, and overhanging at one end. Let A I, Fig. 48, be the beam supported at A and B, and let the weight of the beam be considered as concentrated with the additional loads. Draw the stress diagram, 0 1 2, as in other cases. Commence at A′, and draw A′ C′ D′ E′ F′ G′ I′ parallel to the radiating lines of the stress diagram, the angles occurring on the verticals let fall from the *weights.* If the applied weights are upon the upper side of the beam, and the reaction at B on the lower side, the force through B will not be encountered as we first pass across the beam. There will be one line, parallel to 0–2, to be added; and we return, as it were, below the beam, drawing the line from I′ to the vertical through B, in the reverse direction. Connect A′ and B′ by a closing line, and the equilibrium polygon is complete. A line through 0, parallel to A′ B′, will divide the load line into the two supporting forces, P_1 at A, and P_2 at B.

If we lay off *a l* below A, equal to P_1, and then construct *l c d e*, &c., as was done in Fig. 4, we shall reach a point *b* under B; then, laying off P_2 upwards from *b* to *b′*, we proceed to *g i*, finally closing on the horizontal line when we subtract the last weight. Drawing the horizontal line marked H through 0, we find the bending moment at any point by multiplying H by the ordinate between the equilibrium polygon and the line A′ B′ or

B′ I′ for that point. At K′, there being no ordinate, the product is zero: consequently the beam is not bent at K. As we pass from K′ to B′ and I′, the ordinate, being below the polygon, may be called negative. The bending moment is in the contrary direction over the portion K I from that existing over A K; and it tends to produce convexity on the upper side of the beam, reversing the tension and compression in the fibres. The point K is called a *point of contra-flexure*. The curvature of the beam is shown to an exaggerated scale by the dotted line A L B N.

In case the beam overhangs sufficiently, or is heavily weighted on the portion B I, it may be found that the line from 0, parallel to B′ A′, cuts the load-line vertical above 1. The supporting force at A will then be negative, and the beam must be fastened down at that point to prevent its rising. The reaction at A will, as a tension in that case, be laid off below *a n* in place of above it. In case the beam overhangs both points of support, we may have two points of contra-flexure; but the overhanging portions may be sufficiently weighted to cause convexity upwards over all the intermediate portion, when the line corresponding to A′ B′ will pass entirely below the curve, and there will be no points of contra-flexure. If the two points of support are brought together into one point, and the overhanging portions balance each other, we have a diagram for each portion akin to Fig. 42. It will now be seen why the moment diagrams of Figs. 41 and 42 were drawn on the upper side of the horizontal or closing line.

98. **7th, Beam supported and fixed in Direction at Both Ends, Weight W in Middle.**—The requirement that the beam shall be fixed in direction at its ends necessitates that it shall be so restrained, by being built into masonry or by the application of certain moments, that the tangent to the curve of the bent beam shall remain horizontal at the points of support. The equilibrium polygon for this case will be the two inclined lines of the moment diagram of Fig. 46. The beam will be horizontal at its middle, by reason of symmetry of loading;

and, referring to Fig. 39, we see, that, from the middle of the span to the abutment, the total change of inclination in one direction must balance that in the other; or, since the changes of inclination are proportional to the ordinates to the closing line, we must draw the closing line at such a height above the vertex of the equilibrium polygon, that, from the middle to one support, the area remaining within the polygon shall just equal the area thus formed without the polygon. Since the two halves of the beam are subject to like forces, the closing line will be horizontal; and, if k represents the maximum ordinate to the original polygon, it is plain that the line must be drawn horizontally, at a distance $\frac{1}{2}k$ from the vertex, to satisfy the prescribed conditions.

We then get, by proportion,

$$\tfrac{1}{2}\,\mathrm{W} : \mathrm{H} = \tfrac{1}{2}\,k : \tfrac{1}{4}\,l; \text{ or } \tfrac{1}{2}\,\mathrm{H}k = \tfrac{1}{8}\,\mathrm{W}\,l = \mathrm{M}\ (\text{max.})$$

at either abutment and at the middle, but of contrary signs, as shown by the direction of $\frac{1}{2}k$. We have points of contra-flexure where the closing line cuts the polygon, or at one-fourth the span from either end. At these points will be the greatest slope, but in opposite directions. Hence

$$tan.\ i = \frac{1}{\mathrm{E\,I}} \cdot \tfrac{1}{8}\,\mathrm{W}\,l \cdot \tfrac{1}{2} \cdot \tfrac{1}{4}\,l = \frac{\mathrm{W}\,l^2}{64\,\mathrm{E\,I}}.$$

The deflection of the middle point below either abutment, or the height of the abutment above a tangent at the middle of the span, will be obtained by taking area moments about the abutment, remembering that one area is positive and the other negative, inducing deflections in opposite directions.

$$v = \frac{1}{\mathrm{E\,I}}\left\{\frac{\mathrm{W}\,l^2}{64}\left(\tfrac{1}{4}\,l + \tfrac{2}{3}\cdot\tfrac{1}{4}\,l\right) - \frac{\mathrm{W}\,l^2}{64}\cdot\tfrac{1}{3}\cdot\tfrac{1}{4}\,l\right\} =$$
$$\frac{\mathrm{W}\,l^2}{64\,\mathrm{E\,I}}\cdot\tfrac{1}{4}\,l\left(1 + \tfrac{2}{3} - \tfrac{1}{3}\right) = \frac{\mathrm{W}\,l^3}{192\,\mathrm{E\,I}}.$$

It will be seen that the bending moments on the middle portion of the beam are the same as would be found were points of support placed at the points of contra-flexure, and the outside portions of the beam removed.

99. **8th, Beam fixed at Both Ends, and carrying a Distributed Load.** — If we draw the equilibrium curve, a parabola, Fig. 47, of depth $k = \frac{w\,l^2}{8\,\mathrm{H}}$, as seen in Case 5, whose area is two-thirds of the enclosing rectangle, and superimpose on it a rectangle whose depth is $\frac{2}{3}\,k$, it is evident, that, the portions which cover one another being neglected, the portion of the rectangle outside the parabola must be equal in area to the portion of the parabola outside of the rectangle: hence the closing line is to be drawn horizontally at $\frac{1}{3}\,k$ above the vertex of the parabola. The bending moment at the abutment is, therefore, twice that at the middle of the span; and the latter is one-third of the moment which would have existed had the beam been simply supported at the ends: hence

$$\mathrm{M} \text{ at abutment} = -\tfrac{2}{3} \cdot \tfrac{1}{8}\,w\,l^2 = -\tfrac{1}{12}\,w\,l^2.$$

$$\mathrm{M} \text{ at middle} = \tfrac{1}{3} \cdot \tfrac{1}{8}\,w\,l^2 = \tfrac{1}{24}\,w\,l^2.$$

To find the point of contra-flexure, we must find that abscissa of the parabola whose ordinate from a tangent at the vertex is $\frac{1}{3}\,k$. Calling the distance of the point of contra-flexure from the middle of the span x, we have

$$k : (\tfrac{1}{2}\,l)^2 = \tfrac{1}{3}\,k : x^2; \text{ or } x^2 = \frac{l^2}{12}$$

$$x = \tfrac{1}{2}\,l\,\sqrt{\tfrac{1}{3}}.$$

The point of maximum slope will be at the point of contra-flexure; for, beyond this point, the curve bends the other way. The value will then be

$$\mathit{tan.}\ i = \frac{\mathrm{H}}{\mathrm{E\,I}} \cdot \tfrac{1}{3}\,k \cdot \tfrac{2}{3}\,x = \frac{w\,l^2}{24\,\mathrm{E\,I}} \cdot \tfrac{2}{3}\,x = \frac{w\,l^3}{72\,\mathrm{E\,I}\,\sqrt{3}}.$$

We may easily obtain the deflection by taking the area moments of the original semi-segment of the parabola and of the semi-rectangle, or

$$v = \frac{w\,l^2}{8\,\mathrm{E\,I}}\,(\tfrac{2}{3} \cdot \tfrac{1}{2}\,l \cdot \tfrac{5}{8} \cdot \tfrac{1}{2}\,l - \tfrac{2}{3} \cdot \tfrac{1}{2}\,l \cdot \tfrac{1}{2} \cdot \tfrac{1}{2}\,l) = \frac{w\,l^4}{384\,\mathrm{E\,I}}.$$

100. **9th, Beam fixed at One End, and supported at the Other, with a Single Weight distant *a* from Fixed End.** — The equilibrium polygon for this weight being drawn, Fig.

49, and its depth denoted by k, we recall, from Case 6, that $H\,k = \frac{W\,a\,(l-a)}{l}$. The position of the closing line is required. As there will be no bending moment at the end which is simply supported, the closing line will start from that extremity of the equilibrium polygon, and meet the vertical dropped from the other extremity at a distance to be found, and here denoted by y_o. As the tangent at the fixed end is horizontal, it will always pass through the supported end: hence the summation of the several small deflections from the fixed to the supported end must be zero. Or, as the triangle with y_o for its altitude overlaps the polygon first drawn, we may say that its area moment about the supported end must equal the area moment of the original triangle. We have, then,

$$y_o \cdot \tfrac{1}{2}\,l \cdot \tfrac{2}{3}\,l = \frac{W\,a\,(l-a)}{H\,l} \cdot \tfrac{1}{2}\,l \cdot \tfrac{2}{3}\,\frac{l+l-a}{2} = \frac{W\,a\,(l-a)\,(2\,l-a)}{6\,H}$$

$$y_o = \frac{W\,a}{2\,H\,l^2}\,(l-a)\,(2\,l-a).$$

The bending moment at the fixed end is, therefore,

$$H\,y_o = -\frac{W\,a}{2\,l^2}\,(2\,l^2 - 3\,a\,l + a^2).$$

To determine the point of contra-flexure, we have the condition that the ordinate to the original triangle at the distance x from the fixed end shall equal the ordinate to the closing line at the same point, or

$$\frac{k}{a}\,x = \frac{y_o}{l}\,(l-x)\,; \text{ or } \left(\frac{k}{a} + \frac{y_o}{l}\right) x = y_o\,;$$

$$x = \frac{y_o\,a\,l}{k\,l + a\,y_o};$$

a quantity easily computed when a definite example is attempted. Upon drawing a line in the stress diagram parallel to the closing line, the supporting forces will be determined. The maximum bending moments of opposite signs occur at the weight and the fixed end. The value of the deflection can now be calculated, the point where the beam is horizontal being found first.

101. **10th, Beam fixed at One End, supported at the Other, and uniformly loaded.** — Here the curve being a parabola, Fig. 50, we superimpose a triangle whose area moment about the supported end shall equal that of the parabolic segment, in order that the beam may remain horizontal at the fixed end: therefore

$$y_0 \tfrac{1}{2} l \cdot \tfrac{2}{3} l = \tfrac{1}{8} \frac{w l^2}{\mathrm{H}} \cdot \tfrac{2}{3} l \cdot \tfrac{1}{2} l; \text{ or } y_0 = \frac{w l^2}{8 \mathrm{H}} = k.$$

The bending moment at the fixed end is, then, $\mathrm{M} = -\frac{w l^2}{8}$.

The point of contra-flexure is obtained by (see 9th Case)

$$\frac{y_0}{l}(l - x) = \frac{k}{(\frac{1}{2} l)^2} x (l - x); \; x = \frac{l y_0}{4 k} = \tfrac{1}{4} l.$$

Another point of maximum bending moment will be at the middle of the small parabolic segment; that is, at three-eighths of the span from the supported end: its value may be proved to be

$$\mathrm{M} = \frac{9}{128} w l^2.$$

The beam will be horizontal at the point where the positive area cut off on the left of the ordinate equals the negative area. The deflection will be measured to such point. The reaction at the supported end will be $\frac{3}{8} w l$. It is not expedient to carry out all of these steps in detail here: the method of doing so has already been indicated.

102. **Beam of Two Spans; Special Device.** — The last two cases might have been treated as beams of two equal spans continuous over a pier; for an equal and symmetrical load on each span would have made the beam horizontal over the middle support. The treatment would have been exactly the same. A method of analyzing the last case is here submitted, however, as illustrative of a modified treatment. Suppose that a beam of span l carries a uniform load of w per unit of length: the deflection at the middle, by Case 5, will be $\frac{5}{384} \cdot \frac{w l^4}{\mathbf{E I}}$. If a force from below is applied at the middle of the span, of

just the magnitude required to bring this point to its original position, on a level with the two ends, the beam will be transformed to one of two equal spans, each one-half of the original amount, and the applied force will be the reaction of the middle pier. By Case 4, the deflection from a weight W at the middle of a span is $\frac{W\,l^3}{48\,\mathbf{E\,I}}$. Putting these two deflections equal to one another, we at once obtain $W = \frac{5}{8}\,w\,l$; or, if the new span $l' = \frac{1}{2}\,l$, $W = \frac{5}{4}\,w\,l'$, and the end reactions will each be $\frac{3}{8}\,w\,l'$. If the load line be divided accordingly, the closing lines of the equilibrium polygon can be drawn, and all the values of Case 10 at once obtained.

CHAPTER VII.

CONTINUOUS TRUSS OF TWO SPANS.

103. **General Principle of Continuity.** — The fact which was brought out in the last chapter, that the *area moment* of the equilibrium polygon is proportional to the deflection of a beam or truss, will enable us to deal with continuous trusses easily. A truss extending over two spans will first be taken up. Suppose that we have a beam, represented in Fig. 51, supported on the two abutments A and C, and divided by the pier at B into two unequal parts: its own weight is uniformly distributed, and it carries, in addition, a uniformly distributed load of twice the intensity from C to D, and also from E to F. Divide the two spans into convenient parts, equal or unequal (here taken equal), and consider the load to be concentrated at the points of division. Draw verticals through these points, and having constructed the load line and stress diagram, 0 1 2, draw the equilibrium polygon between C′ and A′ as in previous examples. The loads at A and C are neglected, and are represented by the portions of the load line which project at 2 and 1. Since B carries a load, the vertical through B will determine one of the angles of the polygon in the same way as any other loaded point.

As no bending moment exists at A, the desired closing line A′ B′ must start from A′, and similarly C′ B′ is drawn through C′. We know that B′ must lie below the curve; for we have a negative moment of flexure over B; that is, a moment which makes the beam convex on the upper side. We usually have two points of contra-flexure where A′ B′ and C′ B′ cut the

curve. Some condition is necessary to limit the position of B′, since there is manifestly but one correct value for the moment of flexure over the pier for a given position of the load.

104. **Abutment Deflections.** — From the demonstration in regard to *deflection* which has gone before, it may be seen that the vertical displacement of a point D, Fig. 51, in a beam under flexure, in reference to some point, such as C, as origin, and from a tangent C L to the beam at that origin, depends upon the successive changes of inclination between the two points, and the distances from the point D at which they occur, regard being paid to the direction of the change of inclination. Then, as each change of inclination is proportional to the ordinate to the equilibrium polygon, the deflection of a point from a tangent through the origin is proportional to the summation of the products of each ordinate into its distance from the point in question. As the summation of these products is the same thing as the area between C′ and D′ multiplied by the distance of its centre of gravity horizontally from D′, which we have styled an *area moment*, the deflection of D from the tangent through C is proportional to the area C′ D′ multiplied by the horizontal distance of its centre of gravity from D′. If, then, C L is the tangent through C, the deflections or vertical distances of the points B and A from this tangent, or B K and A L, will be proportional to the proper area moments, and the closing lines to B′ must be so drawn as to satisfy this condition. From the similarity of triangles, B K and A L are proportional to B C and A C, two known quantities.

We may, with advantage, by drawing a tangent M B N to the beam at the pier B, obtain a relation of area moments more easily remembered and used. It is evident that N C : M A = B C : B A. From the preceding reasoning, denoting the centres of gravity of the respective areas by *a*, *b*, *c*, and *d*, and taking the distances towards the opening of the angle, we write a proportion of area moments

$$\frac{\text{Area } C'\,P'\,D' \,.\, d\,l - \text{area } K'\,D'\,B' \,.\, c\,k}{\text{Area } K'\,B'\,G' \,.\, b\,h - \text{area } A'\,F'\,G' \,.\, a\,g} = \frac{C\,N}{A\,M} = \frac{B\,C}{A\,B}.$$

The deflection M A being on the opposite side of the tangent from N C, the similar areas in the above proportion are taken with the opposite signs; that is, K′ D′ B′ being reckoned as negative in the first term of the proportion, K′ B′ G′ is considered positive in the second term; and so of the others. Or we may consider the distances to the right of B plus, and those to the left minus. It is evident that there is but one position of B′ which will satisfy the above condition: for, if B′ is carried still farther below K′, the first term of the proportion is diminished, and the second term is increased; while, if B′ is raised, the reverse takes place. It will be remembered that area moments do not give absolute deflections, but are only proportional to them: to obtain actual deflections, the area moments must be multiplied by H, and divided by **E I**. As these quantities are constant, they disappear from the equation just deduced. If the moment of inertia of the beam or truss is variable, its rate of variation being known, each ordinate must be changed in the due proportion before the areas are computed. Generally **I** is considered to be constant.

105. **Areas and Centres of Gravity.** — The areas in question are easily measured. If they are bounded by broken lines, as is the case when we deal with concentrated loads, the ordinates represented by the dotted lines can be scaled and multiplied by the horizontal distance between them, which is usually constant. As a continuous load gives a curve which passes through the vertices of the polygon described when the same load is concentrated on detached points and the abutments, and as uniformly loaded portions have parts of parabolas for their equilibrium curves, many of the areas are parabolic segments, triangles, and combinations of the two. The centre of gravity of a parabolic segment, such as C′ D′ P′, is at *d*, half way horizontally between C′ and D′. The centre of gravity of any triangle whose base is vertical is on the ordinate drawn at one-third of the horizontal distance from the base to the vertex.

If we connect C′ and K′ by a straight line, the segmental area C′ P′ K′ and the triangle C′ K′ B′ will mutually overlap;

so that $C' P' D' . dl - K' D' B' . ck$ will be equal to $C' P' K'$ multiplied by the distance of its centre of gravity horizontally from C', minus $C' K' B'$ multiplied by two-thirds of the span B C. If an area is partly bounded by portions of two different parabolas, the common point of the two parabolas where the intensity of the load changes may be connected with the extreme points of the area, and it will thus be divided into a triangle and two parabolic segments. It may be unnecessary to add that any area may be divided into a number of parts, the respective centres of gravity found, and then the area moments of these parts calculated and combined, with the same result as if the area had been treated as a whole. $C' P' D'$ is bounded by a broken line; but, as the angles of this line lie on a parabola, the centre of gravity of the area is still half way horizontally between C' and D'.

As, with concentrated loads, areas are made up of trapezoids, the following method of finding the centre of gravity of any four-sided figure may be convenient when great accuracy is desired. Draw the two diagonals $a\,c$ and $b\,d$, Fig. 53; *bisect* one at m, and lay off the short segment $a\,e$ of the other at $c\,f$. Connect f and m, and the centre of gravity of the quadrilateral will lie in the line $f\,m$, at one-third of its length from m, the middle point of the diagonal bisected. By reversing the process with the diagonals, bisecting $a\,c$, and laying off $d\,e$ from b, another line may be drawn: the centre of gravity will be at its intersection with $f\,m$.

106. **Value of the Pier Ordinate** y_o. — The distance $K' B'$ may be determined easily, without the necessity of making trials to ascertain its value. We will illustrate by a simple example: the proof made use of applies to any case. Let a beam of two spans, c and d, Fig. 52, have a single load on each span. The equilibrium polygon will be similar to A C D B, the one represented. Let us suppose for an instant that there is no bending moment over the pier. In that case, drawing A I and I B, we should complete our figure, and calling the area of the triangle $A\,C\,I = \mathbf{A}$, of $IDB = \mathbf{B}$, and the distances

of the centres of gravity of these triangles from the verticals through A and B respectively **a** and **b**, we ought to have, by § 104, if the ends of the beams are on the supports,

$$\frac{-\mathbf{A}\,\mathbf{a}}{\mathbf{B}\,\mathbf{b}}=\frac{c}{d}.$$

Since a bending moment over the pier does exist, this equation will not be true. Then change the lines A I and I B to A E and E B, moving on the vertical a distance I E $= y_o$. The area moments on one side are proportioned to the area moments on the other as c to d; but the area moments on the left are equivalent to

$$\mathbf{A}\,\mathbf{a}-\mathrm{AIE}\,.\,\tfrac{2}{3}\,c,$$

$\frac{2}{3}\,c$ being the distance of the centre of gravity of A I E from the abutment vertical. The area of A I E is $\frac{1}{2}\,c\,y_o$. A similar relation exists on the right. Therefore we may state our proportion as follows: —

$$\frac{-\mathbf{A}\,\mathbf{a}+\frac{1}{2}\,c\,y_o\,.\,\frac{2}{3}\,c}{\mathbf{B}\,\mathbf{b}-\frac{1}{2}\,d\,y_o\,.\,\frac{2}{3}\,d}=\frac{c}{d}.$$

Every thing here being known except y_o, we obtain the distance which in general, as in Fig. 51, B′ should be below K′, when c and d denote the spans, **A** and **B** the areas A′ F′ K′ and K′ P′ C′, and **a** and **b** the distances to the abutments from their respective centres of gravity, —

$$y_o=\frac{3\,(\mathbf{A}\,\mathbf{a}\,.\,d+\mathbf{B}\,\mathbf{b}\,.\,c)}{c\,d\,(c+d)}=\frac{3}{c+d}\left(\frac{\mathbf{A}\,\mathbf{a}}{c}+\frac{\mathbf{B}\,\mathbf{b}}{d}\right).$$

If $c=d$, $$y_o=\frac{3}{2\,c^2}\,(\mathbf{A}\,\mathbf{a}+\mathbf{B}\,\mathbf{b}).$$

There is no failure when the beam happens to be horizontal over the pier B; for then M A and N C are each zero, and therefore area C′ P′ D′ . $d\,l$ = area K′ B′ D′ . $c\,k$, or K′ B′ G′ . $b\,h$ = A′ F′ G′ . $a\,g$.

107. **Remarks.** — The equilibrium polygons of the two spans might be constructed separately, as we should do for detached spans; but in this case we must have the same value of H for both polygons, and they must meet at one point on the vertical

through the centre pier: therefore, in drawing the polygons, we may start from K′, or any other point in the pier vertical, and work each way. Indeed, the polygon might cross a horizontal line through C′, and can be transferred at any time, if desired, to that line, by measuring off ordinates either above or below it, so that A′ B′ and B′ C′ shall coincide with or become the horizontal line. The construction of the polygon for each span, by a separate stress diagram, will be shown a little later in an example of four continuous spans. With a symmetrical load on the two spans, we may introduce the above value of y_o into, and solve, Cases 9 and 10, §§ 100, 101.

108. **Shear Diagram.** — As the bending moments at all points are thus determined in Fig. 51, it remains to discuss the shear diagram. Upon drawing from 0 two lines, 0–4 and 0–5, parallel to B′ C′ and A′ B′, we shall divide the load line into three portions, which are, 1–4 $= P_3$, the supporting force at C, 4–5 $= P_2$, the supporting force at B, and 5–2 $= P_1$, the supporting force at A. Lay off $P_1 = 5–2$ at *m n;* draw the inclined lines, as was done in Fig. 6, steeper where the load is more intense, striking the vertical under the pier at *q;* make $q\,r = P_2 = 4–5$; and complete the diagram by reaching *s* at a distance $t\,s = -P_3$ below *t*. The ordinates between *m t* and the lines just drawn give the shearing forces at all points, on the left of a plane of section and positive when above *m t*.

109. **Discussion.** — If the load is shifted in position, we may draw new equilibrium polygons, and then complete the diagrams; but, as in the case of a single span, a few diagrams will suffice, as will be seen presently.

The beam of Fig. 51, as now loaded, has two points of contra-flexure, — at D and G. It may happen, that, when one of the spans is much more heavily loaded than the other, the point of contra-flexure G′, on the shorter span, moving towards the outer end, may finally pass off altogether. As G′ moves towards A′, the point 5, where the line parallel to B′ A′ cuts the load line, will approach 3; and when G′ reaches A′, and disappears, 5 will pass beyond 3. There will still be some slight pressure on the

abutment, although the span A B will be convex upward throughout its whole extent; and the end of the beam will not rise from the abutment A until it is found necessary, in order to satisfy the condition of proportionality of area moments, to so draw A′ B′ that its parallel line 0–5 passes entirely outside of the end 2 of the load line. As soon as this occurs, unless the beam is fastened down, it must be treated as one resting on two supports, and overhanging at one end, § 97, Fig. 48.

110. Formula for Pier Moment for a Continuous Load.— If the beam is completely covered with a uniform load, the equilibrium curve will be a parabola, and the middle ordinate for one span will be, by § 95, $\frac{w\,l^2}{8\,\mathrm{H}}$. If l_1 and l_2 denote the two spans, w_1 and w_2 the weight of load on the respective spans per unit of length, the formula of § 106 becomes, when we remember that the area of the parabolic segment $= \frac{w\,l^2}{8\,\mathrm{H}} \cdot \frac{2}{3}\,l$,

$$y_o = \frac{3}{l_1 + l_2}\left(\frac{w_1\,l_1^3}{12\,\mathrm{H}\,l_1} \cdot \tfrac{1}{2}\,l_1 + \frac{w_2\,l_2^3}{12\,\mathrm{H}\,l_2} \cdot \tfrac{1}{2}\,l_2\right);$$

or, if $\mathrm{M}_o =$ pier moment,

$$\mathrm{H}\,y_o = \mathrm{M}_o = \frac{w_1\,l_1^3 + w_2\,l_2^3}{8\,(l_1 + l_2)}.$$

If $l_1 = l_2$ and $w_1 = w_2$, we get the result of § 101.

Since the load on a truss is concentrated at joints, the equilibrium polygon for such a complete load will be *inscribed* in the parabola for the same load continuously distributed as on a beam: hence the area of this polygon will be a little less than $\frac{w\,l^3}{12\,\mathrm{H}}$, the area given above. It follows that the bending moment over the pier for a continuous truss of two spans, loaded at joints, is slightly less than the bending moment by the above formula. The original formula for y_o, § 106, will give the correct ordinate.

111. Extent of Load to produce Maximum Moments.— In treating a truss of one span, we found, that, since a load at any point caused positive bending moments at all points of the

span, the maximum bending moment, and hence the maximum chord-stress at every point, would occur when all possible loads were placed on the bridge, or it was covered from end to end. An inspection of Fig. 51, just discussed, will show that positive and negative bending moments occur in different parts of the same span, and that while a load added at any point in the beam already subject to positive bending moment will increase the bending moment, another weight put on near and to the right of B will tend to diminish the negative moment at the point of application by causing the point of contra-flexure to move nearer K′.

Suppose that the beam of Fig. 55 carries a single weight only on the span B C $= l_2$, at a distance a from C. The equilibrium polygon for the span B C will be a triangle, and there will be none for the other span. By § 96 we see that k will equal $\dfrac{\mathrm{W}\, a\,(l_2 - a)}{\mathrm{H}\, l_2}$, and that

$$\mathbf{B} = \frac{\mathrm{W}\, a\,(l_2 - a)}{2\,\mathrm{H}}; \quad \mathbf{b} = \tfrac{1}{3}\,(l_2 + a).$$

Substituting in the equation for y_o, § 106, we get

$$y_o = \frac{3}{l_1 + l_2} \cdot \frac{\mathbf{B}\,\mathbf{b}}{l_2} = \frac{3}{l_1 + l_2} \cdot \frac{\mathrm{W}\, a\,(l_2^2 - a^2)}{6\,\mathrm{H}\, l_2} = \frac{k}{2} \cdot \frac{l_2 + a}{l_1 + l_2}.$$

If, upon plotting this value of y_o, and drawing the closing lines, we find that the weight at a distance a from C causes a *positive* moment over any portion of the beam which has a *negative* moment under a full load, it is evident that the removal of this weight will increase the negative moment by just the amount of the positive moment thus removed from the particular point. The use of this construction will be seen in the sequel.

112. **Example.**—We will make a practical application of this method to a truss of two spans, one of 80 feet, and the other of 100 feet, as represented in Fig. 54, continuous over the pier. Let each truss weigh $2\frac{1}{2}$ tons per panel of ten feet, or 500 pounds per foot; and let the rolling load be a thousand

pounds per foot of one truss, or one ton per foot for the bridge. A panel weight for one truss will be maximum $7\frac{1}{2}$ tons, minimum $2\frac{1}{2}$ tons.

Take 1–2 = 135 tons, the weight of both spans when fully loaded; divide 1–2 into portions of $7\frac{1}{2}$ tons, with end portions at 1 and 2 of $3\frac{3}{4}$ tons at A and C; assume a point 0, preferably opposite the middle of 1–2, and at a distance in this figure of 50 tons. Since the truss is drawn as $12\frac{1}{2}$ feet high, it will simply be necessary to multiply the ordinates to the equilibrium polygons by *four* to obtain the stresses in the chords; and this can be done, without multiplication, by measuring the ordinates by the proper scale, thus converting the moment diagram into a chord-stress diagram at once. Leave out of consideration the end portions of $3\frac{3}{4}$ tons, which come directly upon the abutments, and, commencing at A′, draw the polygon A′ M C′ parallel to the several lines connecting 0 with the points of division of the load line. Only the extreme lines radiating from 0 are drawn in the figure, as the remainder would confuse it, and tend to render the position of the point 0 uncertain. As we know that the middle ordinate of **A** must be $\frac{7\frac{1}{2} \times 10 \times 100}{8 \times 50} = 18\frac{3}{4}$ feet, the remaining ordinates will be easily calculated, 18, $15\frac{3}{4}$, 12, and $6\frac{3}{4}$ feet: hence, summing all of the ordinates, and multiplying by the constant panel length 10 feet, we get $\mathbf{A} = 1237.5$. The other area may be obtained similarly. Scaling from a diagram of reasonable size will answer as well.

Now calculate M B′ by the formula for y_o, § 106, by which

$$\text{M B}' = \frac{3}{100+80}\left(\frac{1237.5 \times 50}{100} + \frac{630 \times 40}{80}\right) = \frac{1}{60}(618.75 + 315) = 15.56 \text{ ft.}$$

(The formula for a *continuous* load, deduced in § 110, would give

$$\text{M B}' = \frac{\text{M}}{50} = \frac{\frac{3}{4} \cdot 80^3 + \frac{3}{4} \cdot 100^3}{50 \cdot 8\,(80+100)} = 15.75 \text{ ft.}$$

The previous value is the correct one.)

Laying off M B′ = 15.56 feet, draw A′ B′ and B′ C′. The figure A′ M C′ B′ encloses the ordinates for bending moment

when both spans are fully loaded. Draw 0–3 and 0–4 parallel to A′ B′ and B′ C′; lay off 2–3 upwards at *a d*, and 4–1 downwards at *c e;* draw *d f* at an inclination of 7½ tons to a panel; make *f g* equal to 3–4; and draw *g e* at the same inclination: it should close on the point *e*, just plotted. The figure *a d f g e c b a* encloses the ordinates for shearing force when both spans are fully loaded. These statements follow from the investigations of single spans.

113. **Load on One Span only.** — Remove all of the rolling load from B C, including in this figure the load on B: the load line will extend from 2 to 5, and the extreme radiating lines are again shown by full lines. Use the equilibrium polygon A′ M, and add the part from M to C″. As the load on B C is one-third of its former amount, **B** will now be 210, and the other quantities will be unchanged: hence the pier ordinate will be

$$\text{M B}'' = \frac{1}{60}(618.75 + 105) = 12.06 \text{ ft.}$$

Plot this value; draw A′ B″ and B″ C″. Find anew the supporting forces; lay off *a h* upwards; draw *h i* at an inclination of 7½ tons to a panel; make *i l* equal the pier reaction plus one-half panel weight of moving load, or 2½ tons; draw *l k* at an inclination of 2½ tons per panel, closing with *k c* for the abutment reaction at C. The half-panel weight of moving load is added at *i*, because the entire travelling load was removed from B, and the real diagram would have been made by a line, shown in the figure, at an inclination of 2½ tons per panel, meeting *i h* in the middle of the first panel from the pier. It is easier to draw the complete line *h i;* and no error will arise, as the shear ordinates are measured in the middle of each panel. If the two abutment reactions are plotted first, we need pay no heed to the half-panel weight of rolling-load on B, as the two inclined lines for shear will intercept between them on the pier vertical the proper reaction plus the half-weight.

114. **Load again shifted.** — If the remaining portion of the rolling load is removed, uncovering the span A B, the load line will be reduced to 5–6. As the part C″ M of the equilib-

rium polygon applies to the second span, add the portion M A″, and, finding the point B″″, complete the figure: M B″″ will be in this case exactly one-third of M B′, because that is the ratio of the two loads. The lines limiting the ordinates in the diagram of shearing force will be *m n* and *p q*. The polygon A″ M C′ will represent the case of the shorter span covered with the rolling load, including the point B, and the longer span unloaded; and, as the value of M B‴ will now be

$$y_o = \tfrac{1}{60}\,(206.25 + 315) = 8.69 \text{ ft.},$$

we draw A″ B‴ and B‴ C′, find the supporting forces, and thence the lines *r s* and *u t*. (See the closing remark of § 113.)

Each one of the equilibrium polygons might be drawn independently; but by the method here carried out, of passing them all through a common point M, two complete polygons suffice for four different positions of moving load; and, in any case of trusses with horizontal chords, but little need be added to these polygons to determine the maximum chord-stresses. The two polygons might have passed through M without coinciding for one panel, as in this figure; in which case B would probably have been relieved of one-half of its rolling load when one span was unloaded. Some readers may prefer such a treatment; but we should then require new divisions of the load line, in place of using the original divisions as here. It is evident that the presence of a load at B, or its absence from that point, can in no way affect the bending moments.

115. **Discussion of Results.** — A little study of the polygons for bending moment will show, that, for the given intensities of load, all possible polygons, if drawn to pass through M, will lie between M A′ and M A″ on one side, and M C′ and M C″ on the other; also that the maximum bending moment which tends to make the truss concave on the upper side occurs when *one span* is *fully loaded*, the other being at the same time without travelling load. This fact might be anticipated, since the addition of a load on one span will tend to increase the deflection of all points on that span, and to diminish the deflec-

tion of all points on the other span. The maximum bending moment over the pier will occur when *both* spans are completely loaded, for every increment of load adds to such negative moment; and its value will, therefore, be H . M B′. This moment will give tension in the top chord, and compression in the bottom chord, at that section. Conversely, M B′′′′ must be the shortest ordinate, as the polygon to which it relates represents the lightest possible load. All values of y_0 will, therefore, lie between M B′ and M B′′′′.

As A′ M C′ and A″ M C″ are the limiting polygons, they give the extreme deviations of the points of contra-flexure; and from the positions of these points can readily be determined the portion of each chord subject to tension alone, to compression alone, and the portion which must be adapted to withstand either stress as the load shifts its position. It will be noticed that the point of contra-flexure for the hundred-foot span shifts from the second panel from B, when only that span carries the moving load, to the fourth panel from B when only the other span is fully loaded; while, for the eighty-foot span, the point of contra-flexure shifts from the second panel from B to the seventh panel from B under similar variations of load. The influence of the longer span on the shorter is very marked, as, when the longer one is fully loaded, the unloaded span presses at C with only the weight represented by $c\,k$. A check on the accuracy of construction is found in the fact, that, when both spans are loaded and both unloaded, the points of contra-flexure occur at the same place.

116. **Length of Chord under each Kind of Stress.** — Since there is always a negative bending moment at the *first* joint on either side of the pier, we shall find, when we take moments at that joint of the bottom chord and again at the first joint in the top chord, that the first panel of the top chord will always be under tension, and that in the bottom chord the second panel from the pier will always be subject to compression. The fine line of the figure denotes a tension, and the heavy line a compression member. The double line on the adjoining portions

of either chord signifies that such pieces must be adapted to resist both tension and compression.

When the point of contra-flexure advances to the fourth panel from the pier in the hundred-foot truss, there will still be a positive bending moment at the joint beyond: hence in the bottom chord the fifth panel from the pier will be a tension member, and in the top chord the fourth panel from the pier will always be in compression. As, in the eighty-foot span, the point of contra-flexure may reach the second panel from C, the first joint from C will be the only one always under a positive bending moment. The supporting force will then be $k\,c$, and the shear line $k\,l$, the shear will be of opposite signs in the first and second panels, the ties which meet at the common joint of the bottom chord for these two panels will be in action, and therefore two panels of the top chord will be in compression. There will be no corresponding stress in the lower member, as the two tie-braces react against one another. The first panel of the bottom chord, of course, has no stress.

117. Partial Load on One Span. — The polygon M D′ on the left applies to the case where the span from A to D has upon it no moving load. If the polygon C″ M E D′ is taken, the moving load extends from B to D only; if C′ M E D′ is used, the load covers the span B C also. In the first case, the required point on the pier vertical is found to be just below B‴, as seen in the figure; and, in the second case, a little below B″. The dotted lines from C″, C′, and D′ to M, and from E to M and D′, show the different areas used in finding the values of y_o. The areas may be scaled or computed, as thought best. Upon finding the reactions, we construct $y\,v\,w$ and the dotted line between $q\,p$ and $k\,l$, as the lines limiting the ordinates for shear when a moving load extends over B D alone; while $x\,z\,o$ and the dotted line between $t\,u$ and $e\,g$ will determine the shear ordinates for a moving load over C D. The inclinations of these lines correspond to the intensities of the loads, and the inclination changes in the middle of the panel at the head of the load.

118. **Completion of Shear Diagram; Analysis.** — The greatest pressure on the abutment A occurs when A B is fully loaded, and B C carries no moving load. The supporting force is then $a\,h$, and the shear in the two spans will be given by ordinates at the middle of each panel to the lines $h\,i$ and $l\,k$. For the reason why ordinates should be measured in the middle of each panel, see § 17. If the load extends over both spans, the supporting force at A falls to $a\,d$, and we get the shear by drawing $d\,f$ and $g\,e$. If neither span has any rolling load upon it, we find the pressure at A to be $a\,m$, and then draw $m\,n$ and $p\,q$. Finally, if B C alone carries the rolling load, the pressure at A diminishes to $a\,r$, and the shear diagram will be completed by the lines $r\,s$ and $u\,t$.

Again: if the rolling load, at first extending entirely from A to C, moves off from the portion A D, the supporting force at A diminishes from $a\,d$ to $a\,x$, and the shear will be given in the span A B, as lately stated, by ordinates to the lines $x\,z\,o$; the point z occurring in the last lightly-loaded panel D. If the rolling load still covers B D, but the load on B C is supposed to be removed, the supporting force at A will immediately increase to $a\,y$, and the bounding lines for the ordinates will now be $y\,v\,w$. In the span B C the inclined dotted line of the greater inclination will give the shear for the first arrangement of load, and the dotted line of less inclination will limit the shear ordinates for the second arrangement of load. As it is manifest that $h\,i$ and $m\,n$ are the limiting lines for the extreme cases of load over A B alone and load upon neither span, and as the load may cover any number of panels from one to ten, from B towards A, the sets of lines, of which $y\,v\,w$ is one, will, for different positions of the load, shift between $h\,i$ and $m\,n$, and the point v, at the head of the load, will move on a curved line from the middle of one end panel to the middle of the other, in the span A B. At the same time, the line for the unloaded span moves parallel to and between $k\,l$ and $p\,q$.

If a load, having covered B C, should then extend from B

towards A, we should obtain in a similar manner a curve, traced by the point z, between the lines $d\,f$ and $r\,s$, from middle to middle of end panels. As the ordinates to this curve above the line $a\,b$ are less than those to $h\,v\,n$, the latter line only need be drawn. Therefore, for a load advancing from B, the ordinates to $h\,v\,n$ at the middle of each panel determine the maximum shear of this kind. Although this curve is not exactly a parabola, the construction of § 20 will give a curve which comes very close to the actual one, agreeing at the ends, and giving shear ordinates slightly in excess of the truth at the middle portion of the span. As the shear curve extends between the middle ordinates of the first and last panels of the span, set off the half-panel from h and n, and then divide the remaining portion of each tangent, up to their common intersection, into a number of equal parts one less than the number of panels in the span. The divisions are marked in this figure from p to t. While the small error is on the side of safety, the amount which the constructed curve passes outside of the true point v will indicate the magnitude of the greatest deviation; and the curve can then be corrected by setting in, towards its tangents, as is usual in corrections, according to the square of the distance. The allowance can be easily made by the eye, and is seldom large enough to be of practical importance. In any case, by placing a load on half of the span, and determining a point similar to v, the curve can be located with all desired accuracy.

If a rolling load advances from A, the span B C being unloaded, the pressure at A increases from $a\,m$ to $a\,h$, and the different limiting lines for shear will move between $h\,i$ and $m\,n$; but as the inclinations of the lines which correspond to $y\,v$ and $v\,w$ will be reversed, since A D now is loaded, and D B is not, the point j at the angle will, like v, trace a curve from the middle of the panel near m to the one near i of the opposite curvature. If, again, B C is loaded, and a load advances from A towards B, the curve drawn in the figure from r to f will be described. As the latter curve includes the former, it

is the only one required. While the load is shifted as just described, the line for the other span vibrates between the parallels $k\,l$ and $p\,q$ in the one case, or $t\,u$ and $e\,g$ in the other.

The shear-curves $h\,v\,n$ and $r\,j\,f$ limit all the ordinates for shearing force in the span A B for every position of moving load; the portion of the curve $h\,v\,n$, which lies above $a\,b$, giving the maximum shearing stress upwards on the left of any section, and calling for diagonals all sloping one way from A to the panel over v. The truss of the figure has ties. The part of $r\,j\,f$ which lies below $a\,b$ gives the maximum downward shear on the left of a section, and requires braces from B to the third panel from A, as shown, with an inclination in the opposite direction. A similar construction supplies the required shear-curves for the span B C; and, because the spans are dissimilar, all four curves must be drawn, and all possible movements of the travelling load are then provided for. While some may think that a rolling load will never take all of the positions assumed above, we repeat that the worst possible combinations are provided for. It will be seen how the counter-bracing is shifted from the middle of the spans towards the free ends. The stresses in the diagonals can now be obtained by drawing lines parallel to those pieces, as in Fig. 11.

119. **Remarks.** — The desire to have the diagrams clear, while they are on so small a scale, forbids the drawing of many equilibrium polygons and lines of shear for various positions of the moving load; but if the reader will construct a diagram for himself, with the load shifted, panel by panel, for a few panels, he will be able thoroughly to assure himself of the truth of the statements here made as to the limiting values of bending moment and shearing force. As the truss has parallel chords, it is just as easy to have the ordinates to the equilibrium polygon represent chord-stresses as bending moments. It will only be necessary that the middle ordinate at E, for instance, to the dotted line M A′, shall equal $\frac{W''\,l}{8\,k}$, as explained in § 28, and similarly for the other span.

120. Checks on the Accuracy of Diagrams. — There are several tests for proving the accuracy of a set of diagrams. The shear for a particular load should always change its sign at the point where the bending moment for that load is greatest: thus, above the point where $h\,i$ cuts the horizontal line $a\,b$, will be found, if no error has been made, the maximum ordinate of the corresponding moment diagram. The two lines $m\,n$ and $d\,f$ must intersect on $a\,b$, as they apply to the cases of uniform loads over both spans; and $p\,q$ intersects $g\,e$ on $b\,c$ for the same reason. Points of contra-flexure for similar distributions of load must agree. The intersection of $u\,t$ with $p\,q$ must be vertically under the intersection of $g\,e$ with $k\,l$, or $k\,q$ must equal $e\,t$; and similarly $h\,d$ must equal $m\,r$; for a certain weight on one span will diminish the reaction at the farther abutment of the other span a certain definite amount, no matter whether the second span be loaded or unloaded: hence, putting the rolling load over the span B C will diminish the reaction at A from h to d, or an equal amount from m to r, depending upon whether A B is loaded or unloaded. This last check is an excellent one.

121. Maximum Negative Moments. — Since the maximum negative moments at different joints of these spans occur, with one exception, as seen in Fig. 54, when the rolling load is removed from the span in which the joint is situated, there is only one joint to which the method of § 111, Fig. 55, need be applied. At the first joint to the left of pier B it will be seen that the maximum negative moment of Fig. 54 is caused by a complete load on both of the spans A B and B C. The removal of such weights of moving load as cause positive moments at the joint in question will increase the negative moment by the amount of the positive moments removed. In applying the formula of § 111, $W = 5$ tons, $H = 50$ tons, $l_2 = 100$ ft., $l_1 = 80$ ft., and $a = 90$ ft., 80 ft., &c., for successive joints measured from A. For the first joint on the left of B we therefore have, if the load is on that joint,

$$k = \frac{W\,a\,(l_2 - a)}{H\,l_2} = \frac{5 \times 90 \times 10}{50 \times 100} = 0.9 \text{ ft.}$$

$$y_o = \tfrac{1}{2}\,k\,\frac{l_2 + a}{l_1 + l_2} = \frac{0.9}{2} \cdot \frac{190}{180} = 0.475 \text{ ft.}$$

The vertical distance to the closing line at the first joint will be $\frac{9}{10} \times 0.475 = 0.4275$ feet, and the remaining ordinate is $k - 0.427$ feet $= 0.473$ feet.

If the load is on the second joint from B,

$$k = \frac{5 \times 80 \times 20}{50 \times 100} = 1.6 \text{ ft., and } y_o = \frac{1.6}{2} \cdot \frac{180}{180} = 0.800 \text{ ft.}$$

At the first joint the original ordinate will be $\frac{1}{2}\,k = 0.8$ feet, and $\frac{9}{10}\,y_o = 0.72$ feet. The remaining ordinate $= 0.8 - 0.72 = 0.08$ feet.

For a load on the third joint,

$$k = \frac{5 \times 70 \times 30}{50 \times 100} = 2.1 \text{ ft., and } y_o = \frac{2.1}{2} \cdot \frac{170}{180} = 0.992 \text{ ft.}$$

At the first joint, original ordinate $= 0.7$ feet, and $\frac{9}{10}\,y_o = 0.893$ feet. The remaining ordinate $= 0.7 - 0.893 = -\,0.193$ feet.

It will be unnecessary to proceed any further since this load causes a negative moment at the first joint. Adding the previous ordinates for the first joint, and multiplying by H, we get

$$(0.473 + .08)\,50 = 27.65 \text{ ft. tons};$$

which is the amount of negative bending moment to be added to the quantity indicated at the first joint to the left of B, Fig. 54. If preferred, the diagrams may be plotted, as at the bottom of the figure, to a large scale.

As the total ordinate just obtained is about one-half foot, it is evident that no removal of weights from the span B C will increase the ordinate given in the figure at the first joint on the right of B for a complete load on both spans sufficiently to make it greater than the one indicated for steady load alone on B C.

122. Closing Remarks.—While the full lines in this figure

show all of the polygons and diagrams required for a complete discussion of a two-span continuous girder with horizontal chords, a truss with curved or inclined chords cannot be analyzed without an equilibrium polygon for each position of the load, to be combined with the shear diagram, as explained in § 63. Continuous girders are, however, almost always designed with a constant height. Where the height varies, it is proper that the rate at which such a variation in height changes the moment of inertia of the cross-section of the truss should be introduced in the process of equating *area moments*. As such a process is simply finding deflections from a tangent at one of the points of support, **I** was dropped from the denominator as constant. If its rate of variation is assumed, the successive ordinates must be lengthened or shortened, as the case may be, at that rate, before the area is computed, and its centre of gravity found.

If the two spans are equal, the number of lines becomes less as B'' and B''' will coincide, and the shear diagram need be drawn for one span only.

CHAPTER VIII.

CONTINUOUS TRUSS OF MANY SPANS.

123. **Truss of Four Spans; General Treatment.** — From the example presented in Fig. 56 we can deduce such expressions for pier ordinates as will be applicable to any number of spans. This figure shows four spans, of eighty feet, one hundred feet, fifty feet, and forty feet successively, loaded, as represented, with a travelling load, from L to M and N to P, of five tons per panel, and steady load throughout, from A to E, of two tons and a half per panel. To prevent the load line and stress diagram from occupying too much space, as well as to show the perfect practicability of drawing separate moment polygons for each span, as we do for single trusses, we have laid off the load lines of the longer spans independently, but have taken the same value for H in all of the diagrams; and this must always be done. The stress diagram for the eighty-foot span is 0 1 2: the marks of division show the weights on the respective panel joints.

Starting from A′, draw the moment polygon for this span terminating at B′; from B′ draw the moment polygon for the hundred-foot span by lines parallel to those which would complete the stress diagram 0′ 3 4, 3–4 being the load on this span, and the polygon ending at C′. As the last two spans are short, draw 5–6 equal to the load on both spans, and then construct C′ D′ E′, as has been done before. Now draw A′ B′, B′ C′, C′ D′, and D′ E′. Compute the areas between each of these straight lines and the respective moment polygons, and determine the centre of gravity of each area. Areas belonging

to spans which are partially loaded may be divided, as explained in § 105. Let the spans, commencing with A B on the left, be l_a, l_b, l_c, l_d. Let the areas be represented by **A, B, C,** and **D**. Let the distance of the centre of gravity of **A** horizontally from A be **a**, and from B be **a′**; let **b** and **b′**, **c** and **c′**, **d** and **d′**, denote the similar distances for **B, C,** and **D**, from the pier verticals on their left and right.

124. **Pier Ordinates.** — We now desire to find, in the same way as we determined y_o for two spans, the distances y_1, y_2, and y_3, or B′ F, C′ G, and D′ I, the ordinates at the piers, required to complete the diagram for bending moments. In the first place imagine that y_3, or D′ I, is plotted below D′, so that the pier ordinates are all of one sign. If a tangent is drawn at B to the curve which the straight line A B C will assume under the given load, we know by §§ 104, 106, that the deflections at A and C from the tangent at B are proportional to the spans, or

$$\frac{\mathbf{A\,a} - \frac{1}{2} y_1 \cdot \frac{2}{3} l_a^2}{\mathrm{B'\,F\,C'} \cdot \frac{2}{3} l_b + \mathrm{F\,C'\,G} \cdot \frac{1}{3} l_b - \mathbf{B\,b'}}, \text{ or } \frac{\mathbf{A\,a} - \frac{1}{3} y_1 l_a^2}{\frac{2\,y_1 + y_2}{6} l_b^2 - \mathbf{B\,b'}} = \frac{l_a}{l_b}.$$

Similarly for the spans B C and C D when a tangent is drawn through C, and for C D and D E in reference to a tangent at D,

$$\frac{\mathbf{B\,b} - \frac{1}{2} y_1 \cdot \frac{1}{3} l_b^2 - \frac{1}{2} y_2 \cdot \frac{2}{3} l_b^2}{\frac{1}{2} y_2 \cdot \frac{2}{3} l_c^2 + \frac{1}{2} y_3 \cdot \frac{1}{3} l_c^2 - \mathbf{C\,c'}}, \text{ or } \frac{\mathbf{B\,b} - \frac{y_1 + 2\,y_2}{6} l_b^2}{\frac{2\,y_2 + y_3}{6} l_c^2 - \mathbf{C\,c'}} = \frac{l_b}{l_c};$$

$$\frac{\mathbf{C\,c} - \frac{1}{2} y_2 \cdot \frac{1}{3} l_c^2 - \frac{1}{2} y_3 \cdot \frac{2}{3} l_c^2}{\frac{1}{2} y_3 \cdot \frac{2}{3} l_d^2 - \mathbf{D\,d'}}, \text{ or } \frac{\mathbf{C\,c} - \frac{y_2 + 2\,y_3}{6} l_c^2}{\frac{1}{3} y_3 l_d^2 - \mathbf{D\,d'}} = \frac{l_c}{l_d}.$$

Upon reducing these equations, by bringing the unknown quantities together, we obtain

$$2\,(l_a + l_b)\,y_1 + l_b\,y_2 = 6\left(\frac{\mathbf{A\,a}}{l_a} + \frac{\mathbf{B\,b'}}{l_b}\right); \quad (1)$$

$$l_b\,y_1 + 2\,(l_b + l_c)\,y_2 + l_c\,y_3 = 6\left(\frac{\mathbf{B\,b}}{l_b} + \frac{\mathbf{C\,c'}}{l_c}\right); \quad (2)$$

$$l_c\,y_2 + 2\,(l_c + l_d)\,y_3 = 6\left(\frac{\mathbf{C\,c}}{l_c} + \frac{\mathbf{D\,d'}}{l_d}\right). \quad (3)$$

It is evident that (2) is the general equation which applies to any pier of a continuous girder, it having been derived for the pier at C, and the spans B C and C D being restrained at B and D by moments caused by other spans beyond. If, in (2), y_1 equal zero, we have the form of (1); if y_3 equal zero, we get (3); if y_1 and y_3 each equal zero, we get the equation of § 106 for y_o for two spans only. Hence, by beginning with an equation like (1), closing with one like (3), and writing equations similar to (2) a sufficient number of times to make in all one less equation than the number of spans, the required equations for the unknown pier moments for any number of continuous spans will be obtained. If $l_a = l_b = l_c$, &c., the equations simplify a little. The simplicity and symmetry of the general equation (2) is worth noting.

125. **Solution of Equations.** — To show that no difficulty exists in the solution of these equations, let us apply them to the above case, and, since the second members are known quantities, denote them by P, Q, and R. Then

$$
\begin{aligned}
360\,y_1 + 100\,y_2 \qquad\qquad &= \mathrm{P}. \\
100\,y_1 + 300\,y_2 + \ \ 50\,y_3 &= \mathrm{Q}. \\
50\,y_2 + 180\,y_3 &= \mathrm{R}. \quad \text{(Multiply 2d equation by 3.6.)} \\
\hline
360\,y_1 + 1080\,y_2 + 180\,y_3 &= 3.6\ \mathrm{Q}. \quad \text{(Subtract 1st and 3d equations.)} \\
930\,y_2 &= 3.6\ \mathrm{Q} - \mathrm{P} - \mathrm{R}. \\
y_2 &= \frac{3.6\ \mathrm{Q} - \mathrm{P} - \mathrm{R}}{930}.
\end{aligned}
$$

Substitute in first equation, and find y_1; then in third, and find y_3.

126. **Positive Pier Moment.** — If the reader will take the trouble to make a numerical solution of this example with the given intensities and distribution of the load as shown in Fig. 56, he will meet with one peculiar result, — the value of y_3 will be found to be *minus;* that is, D′ I must be measured off from D′ upwards, and denotes a positive bending moment, the truss being concave upwards. There is, therefore, no point of contraflexure in the span D E, and no negative moment over the pier D. The truss presses on the pier D, however, to an amount to

be found presently. That this pressure is small arises from the fact that the load on the hundred-foot span has a tendency to lift the truss from D, and it will be seen that almost all of C D is under a negative bending moment. If the point I had fallen on the straight line joining G with E′, the bridge would have been lifted entirely clear of the pier D; and, if I had come above a line from G to E′, another solution would have been required, with the pier D considered as removed, making C E one span; or else the truss would have required bolting down at D.

127. **Shear Diagram.** — Having drawn A′ F, F G, G I, and I E′, draw in the respective stress diagrams lines 0–7, 0′–8, 0″–9, and 0″–10, parallel to them, cutting off the supporting forces at the several piers arising from each truss, and then complete the diagrams for shearing force which are drawn below the truss. We see that by drawing the stress diagram for each span by itself, as has here been done, we get the supporting force at any pier common to two spans in two portions, one belonging to each span. Thus 7–1 $= a\,p$, and 2–7 $= f\,b$; 8–3 gives $b\,m$, and 4–8 gives $g\,c$. The pressure at B is $m\,b + b\,f$. The pressure at D is $-i\,d + d\,k$ or $i\,k$. The shear diagram is reduced in scale to save room.

128. **Clapeyron's Formula for Uniform Loads.** — If a beam of uniform cross-section is continuous over several spans, and each span is loaded throughout its extent with a load of uniform intensity, such intensity differing on different spans, the area of the equilibrium polygon for any one span, if it is unconnected with the others, will be, by § 95, $\frac{2}{3}\,l\,k = \frac{w\,l^3}{12\text{ H}}$, and the horizontal distance of the centre of gravity from either support will be $\frac{1}{2}\,l$. If, then, these values of **B b** and **C c′** are substituted in (2), § 124, for such a case we get

$$l_b\,y_1 + 2\,(l_b + l_c)\,y_2 + l_c\,y_3 = 6\left(\frac{w_b\,l_b^3}{24\text{ H}} + \frac{w_c\,l_c^3}{24\text{ H}}\right).$$

Multiplying by H, and remembering that H $y =$ moment at pier $=$ M, we have

$$\text{M}_1\,l_b + 2\,\text{M}_2\,(l_b + l_c) + \text{M}_3\,l_c = \tfrac{1}{4}\,(w_b\,l_b^3 + w_c\,l_c^3).$$

The above equation is known as *Clapeyron's Formula*, or the *Three-Mo-*

ment Theorem, as applied to uniform loads on a continuous girder of uniform cross-section. From it may be readily deduced the formula for two spans; and the remark that concentrated loads give slightly less moments over the piers than does the same amount of distributed load applies here also.

129. **Three-Moment Theorem for Single Weight.** — Referring again to the general equation (2), § 124, we see that it is concerned with two spans only, l_b and l_c. If, then, a single load is placed on one of these spans, it will be interesting to see the form which the equation will assume. Suppose that a weight W is placed on l_b at a distance $n\ l_b$ from the left pier, where n denotes some fraction less than unity. Turning to § 96, or looking at Fig. 45, we see, that, for an independent span, the supporting force at the right will be n W; and hence the altitude k of the triangle which represents the area **B**, and whose base is l_b, will be found by the proportion

$$\mathrm{H} : n\ \mathrm{W} = (1 - n)\ l_b : k;\ \therefore k = \frac{(1-n)\ n\ \mathrm{W}\ l_b}{\mathrm{H}}.$$

The centre of gravity of this triangle being found at two-thirds of the distance from the vertex on the line which runs thence to the middle point of the opposite side, the horizontal distance $\mathbf{b} = \frac{2}{3} \cdot \frac{1}{2}\ (1 + n)\ l_b$. Substituting in (2), multiplying by H, and remembering that $\mathbf{C} = 0$, we find that

$$\mathrm{M}_1\ l_b + 2\ \mathrm{M}_2\ (l_b + l_c) + \mathrm{M}_3\ l_c = n\ (1 - n)\ (1 + n)\ \mathrm{W}\ l_b^2 = (n - n^3)\ \mathrm{W}\ l_b^2. \quad (1)$$

If, on the other hand, the weight is placed on l_c at a distance $n\ l_c$ from the left, **B** will be zero, **C** will have the same form as **B** had for the other span; but **c**′, being measured in the other direction, is

$$\mathbf{c}' = \frac{2}{3} \cdot \frac{1}{2}\ \{l_c + (1 - n)\ l_c\} = \frac{1}{3}\ (2 - n)\ l_c:$$

hence we now get

$$\mathrm{M}_1\ l_b + 2\ \mathrm{M}_2\ (l_b + l_c) + \mathrm{M}_3\ l_c = n\ (1 - n)\ (2 - n)\ \mathrm{W}\ l_c^2 = \\ (2\ n - 3\ n^2 + n^3)\ \mathrm{W}\ l_c^2. \quad (2)$$

Quite a practicable way to analyze a continuous girder, although rather a long process, is by putting a weight at one joint of one span, finding the pier moments and reactions at all of the supporting points, then shifting the weight to the next joint, repeating the operations, and so on. The worst combinations of load for each piece can then be selected from a table giving the effect in detail of each weight. The equations for any number of spans will become

$$\begin{gathered}
\mathrm{M}_1\ l_1\ (= 0) + 2\ \mathrm{M}_2\ (l_1 + l_2) + \mathrm{M}_3\ l_2 = 0 \\
\mathrm{M}_2\ l_2 + 2\ \mathrm{M}_3\ (l_2 + l_3) + \mathrm{M}_4\ l_3 = 0 \\
\cdots\cdots\cdots \\
\mathrm{M}_m\ l_m + 2\ \mathrm{M}_{m+1}\ (l_m + l_{m+1}) + \mathrm{M}_{m+2}\ l_{m+1} = (2\ n - 3\ n^2 + n^3)\ \mathrm{W}\ l^2_{m+1} \\
\mathrm{M}_{m+1}\ l_{m+1} + 2\ \mathrm{M}_{m+2}\ (l_{m+1} + l_{m+2}) + \mathrm{M}_{m+3}\ l_{m+2} = (n - n^3)\ \mathrm{W}\ l^2_{m+1} \\
\cdots\cdots\cdots \\
\mathrm{M}_r\ l_r + 2\ \mathrm{M}_{r+1}\ (l_r + l_{r+1}) + \mathrm{M}_{r+2}\ l_{r+1}\ (= 0) = 0.
\end{gathered}$$

The equations for the two piers which carry the loaded span will have a term in W: all the others will equal zero, the end moments will be zero, and there will be one less equation than the number of spans.

For a continuous girder of a large number of spans, the solution of the equations become tedious: the method by *undetermined multipliers* will, perhaps, be the easiest. (We should prefer equations for complete loads, which will give all necessary pier moments.)

130. **Piers not on the Same Horizontal Line.** — Taking up anew the original equation (2), § 124, and multiplying by H, we have

$$M_1 l_b + 2 M_2 (l_b + l_c) + M_3 l_c = 6 H \left(\frac{\mathbf{B\,b}}{l_b} + \frac{\mathbf{C\,c'}}{l_c}\right).$$

It was shown in §§ 88, 104, that **B b** was proportional to the deflection of one of the points of support of a span from a tangent to the beam drawn through the other point of support, and that the absolute deflection in inches would be $\frac{\mathbf{B\,b} \cdot H}{\mathbf{E\,I}}$. If, then, one pier is an amount v vertically below a horizontal line through the top of the other pier at which the tangent is drawn, so much deflection will not be needed to produce the required result when the tangent passes below the horizontal line at that pier, and more deflection will be required if the tangent runs above the horizontal line: that is, imagining the span to first coincide with the inclined tangent, the truss need not be bent so much to place the free end on the other pier, if both the pier and the end of the span are on the same side of the horizontal line, as will be the case when both piers are on a level, or when the tangent and the top of the pier are on opposite sides of the horizontal line.

As we must, however, reckon from the horizontal line when the deflections are made proportional to the spans, the quantity v must be added to the numerator or denominator of the second equation of § 124, according as one pier or the other is taken as the origin. To make v commensurable with **B b** and **C c′**, it must, as seen above, be multiplied by $\frac{\mathbf{E\,I}}{H}$: hence, when intro-

duced in (2), § 124, the equation with which the present section opens becomes

$$M_1 l_b + 2 M_2 (l_b + l_c) + M_3 l_c = 6 \left\{ H \left(\frac{B b}{l_b} + \frac{C c}{l_c} \right) + E I \left(\frac{v_b}{l_b} + \frac{v_c}{l_c} \right) \right\},$$

where v_b and v_c denote the distances the piers at the ends of the spans l_b and l_c are *below* the middle pier. The above equation is the most general form of the Three-Moment Theorem for a girder of constant cross-section, on supports at any elevations, and loaded in any manner. Generally the piers are assumed to be on a level, or rather it is presumed that the wall-plates will remain at those elevations at which they were when the spans on the false works were first made continuous. A small settling of one support will make a serious disturbance of the moments, points of contra-flexure, and reactions. This point will be investigated later (§ 137).

131. **Example: Three Spans.**—It will probably be satisfactory to take up a special example, and go through with the necessary constructions. Let the iron truss, Fig. 57, of three continuous spans, be intended for a railroad bridge: its total length is 624 feet from centre to centre of end pins, and the spans are 192 feet, 240 feet, and 192 feet successively. The dead load is assumed as 1,500 pounds per foot of bridge, and the live load as 2,500 pounds per running foot. The panels are 12 feet long, giving 16 panels for the end spans, and 20 panels for the middle span. On one joint of one truss the dead load is $4\frac{1}{2}$ tons, the live load $7\frac{1}{2}$ tons; total, 12 tons. The trusses are drawn to a scale of 72 feet to an inch, and the diagrams to a scale of 72 tons to an inch.

The load lines for the middle span and for one end span having been drawn as usual, the polygons for bending moment are so drawn as to have their terminal points in the same horizontal line, by which construction we shall diminish errors due to instrumental work, keep the drawing from spreading over the sheet, and make two sets of curves suffice for the discussion. The equilibrium polygons of greater depth are drawn for live and dead load over the whole span; those of less depth,

for dead load only. As a check on the drawing, apply the customary formula, middle ordinate $= \frac{W\,l}{8\,H}$. Here $H = 50$ tons: hence

First span, middle ordinates $= 34.56$ and 92.16 ft.
Second span, middle ordinates $= 54$ and 144 ft.

The two middle ordinates in each span will have the ratio $4\frac{1}{2}$ to 12. The polygons can be constructed by § 28, if desired; and all the ordinates can be easily calculated by the property that they are proportional to the product of the two segments into which each divides the span. Denoting the areas for full load by **A**, **B**, and **C**, and those for light load by **A**′, **B**′, and **C**,′ we have by measurement or calculation (see page 149),

A = 11,736 sq. ft.	A′ = 4,401 sq ft.	A to A′ as 12 to $4\frac{1}{2}$.
B = 22,992 "	B′ = 8,622 "	
C = 11,736 "	C′ = 4,401 "	

Since the centres of gravity of the areas are in the verticals at the middle of each span, the formulæ for pier ordinates become

$$2\,(l_a + l_b)\,y_1 + l_b\,y_2 = 3\,(\mathbf{A} + \mathbf{B}) = 864\,y_1 + 240\,y_2,$$
$$l_b\,y_1 + 2\,(l_b + l_c)\,y_2 = 3\,(\mathbf{B} + \mathbf{C}) = 240\,y_1 + 864\,y_2:$$

whence, by adding and subtracting,

$$\tfrac{1}{2}\,(y_1 + y_2) = \tfrac{1}{736}\,(\mathbf{A} + 2\,\mathbf{B} + \mathbf{C}) \quad (1)$$
$$\tfrac{1}{2}\,(y_1 - y_2) = \tfrac{1}{416}\,(\mathbf{A} - \mathbf{C}). \quad (2)$$

After substituting the proper values of these areas, the sum of the last two equations will determine y_1: their difference will give y_2.

For finding maximum moments we may have six cases, as follows: —

1°. All spans loaded. — $\mathbf{A} = 11736 = \mathbf{C}$; $\mathbf{B} = 22992$: therefore

$$y_1 = y_2 = 94.37 \text{ ft.}$$

2°. All spans unloaded. —

$$y_1 = y_2 = \tfrac{4.5}{12} \times 94.37 = 35.39 \text{ ft.}$$

3°. Middle span loaded, end spans unloaded. — $\mathbf{A}' = 4401 = \mathbf{C}'$; $\mathbf{B} = 22992$.

$$y_1 = y_2 = 74.44 \text{ ft.}$$

4°. End spans loaded, middle span unloaded. — $\mathbf{A} = 11736 = \mathbf{C}$; $\mathbf{B}' = 8622$.

$$y_1 = y_2 = 55.32 \text{ ft.} \quad \text{Or } 94.37 + 35.39 - 74.44 = 55.32.$$

5°. First span loaded, second and third spans unloaded. —

$\mathbf{A} = 11736$, $\mathbf{B}' = 8622$, $\mathbf{C}' = 4401$. $\frac{1}{2}(y_1 + y_2) = 45.36$; $\frac{1}{2}(y_1 - y_2) = 17.63$,

$$y_1 = 62.99 \text{ ft.}; \quad y_2 = 27.73 \text{ ft.}$$

6°. First and second spans loaded, third span unloaded. —

$\mathbf{A} = 11736$, $\mathbf{B} = 22992$, $\mathbf{C}' = 4401$. $\frac{1}{2}(y_1 + y_2) = 84.40$; $\frac{1}{2}(y_1 - y_2) = 17.63$.

$$y_1 = 102.03 \text{ ft.}; \quad y_2 = 66.77 \text{ ft.}$$

These ordinates are plotted below the respective piers, their extremities are connected by straight lines with the ends of the polygons below the abutments, and with each other, as seen in Fig. 57, when the moment diagrams for the given distribution of load are complete. For convenience of reference, the arrangement of loads is marked on each set of lines. By drawing lines in the stress diagrams parallel to the above-mentioned lines we determine the supporting forces, or the abutment and pier maximum and minimum ordinates for the shear diagram.

132. **Points of Contra-flexure; Chord-Stresses.** — The points of contra-flexure are marked by small circles. For the end spans the third case of loading carries these points nearest the abutments, and the fourth case brings them nearest to the piers. For the middle span the fourth case removes the points of contra-flexure altogether, putting the entire top chord into tension, and the bottom chord into compression; while the sixth case carries one point of contra-flexure nearest a pier, thus giving the greatest range of positive moment. It is to be remembered, that, the end spans being alike, what is true of one is true of the other, and that a reversal of the sixth case will carry the points of contra-flexure in the middle span to the same distances on the other side of the centre. We can now, as in Fig. 54, show the portions of either chord which are liable

to one or both stresses. The parallelism of pairs of closing lines which cross the middle span is noticeable.

Those portions of the verticals let fall from the panel joints, which are intercepted between the respective equilibrium polygons and their closing lines, are the distances which alone are significant as giving, or being proportional to, the chord-stresses, and only the longest one of each kind for each joint is valuable. These maximum ordinates have been drawn with full lines, for convenience of reference. It will be seen that the greatest positive ordinates in the first span are given by A F and the greater polygon. The greatest negative ordinate at the pier is B L; and the greatest negative moments for one joint on the left and two joints on the right are given by lines from L, and by the greater polygons. The remaining negative moments for the first span are given by A I and the smaller polygon. In the second span, when it alone carries a complete live load, we have the greatest positive ordinates for the middle joint and five other joints each way from the middle. The sixth and seventh joints from the middle on the right will have greater ordinates to L Q; and by symmetry, when the second and third spans are loaded, the corresponding joints on the other side of the middle will have the same ordinates. The third and fourth joints from the pier B will have maximum negative ordinates between the smaller polygon and G N: for the remaining joints to the middle the ordinates will be measured to F P.

133. **Joints requiring Special Treatment.** — As previously stated, one joint to the left of B, and two joints to the right of B, have maximum negative moments when loads are on the first and second spans: according to §§ 111, 121, since rolling load is upon these spans at the time, so many joints must be unloaded as will thereby influence favorably the increase of negative moments at the above joints. The original formulæ are

$$2\,(l_a + l_b)\,y_1 + l_b\,y_2 = 6\left(\frac{\mathbf{A\,a}}{l_a} + \frac{\mathbf{B\,b'}}{l_b}\right),$$

$$l_b\,y_1 + 2\,(l_b + l_c)\,y_2 = 6\left(\frac{\mathbf{B\,b}}{l_b} + \frac{\mathbf{C\,c'}}{l_c}\right);$$

and the calculations may be made from these formulæ. In that case, as the loaded point is in but one span, two of the areas equal zero. By reference to § 121, the other quantities are easily obtained; but we may more conveniently avail ourselves of equations (1) and (2) of § 129, since they have already been deduced. They must, of course, be divided by H to give y_1 and y_2. For a weight of $7\frac{1}{2}$ tons in the first span of our example, they become

$$864\ y_1 + 240\ y_2 = (n - n^3)\frac{W}{H}(192)^2 = 5529.6\ (n - n^3)$$
$$240\ y_1 + 864\ y_2 = 0.$$

Then

$$\tfrac{1}{2}(y_1 + y_2) = 2.5\ (n - n^3)$$
$$\tfrac{1}{2}(y_1 - y_2) = 4.43\ (n - n^3)$$
$$y_1 = 6.93\ (n - n^3).$$

The value of y_2 is not needed for the weight on the first span.

For a weight of $7\frac{1}{2}$ tons on the second span we also get

$$864\ y_1 + 240\ y_2 = (2\ n - 3\ n^2 + n^3)\frac{W}{H}(240)^2 = 8640\ (2\ n - 3\ n^2 + n^3)$$

$$240\ y_1 + 864\ y_2 = (n - n^3)\frac{W}{H}(240)^2 = 8640\ (n - n^3)$$

$$\tfrac{1}{2}(y_1 + y_2) = 11.74\ (n - n^2)$$
$$\tfrac{1}{2}(y_1 - y_2) = 6.92\ (n - 3\ n^2 + 2\ n^3)$$
$$y_1 = 18.66\ (n - n^2) - 13.84\ (n^2 - n^3);\ y_2 = 4.82\ (n - n^2) + 13.84\ (n^2 - n^3).$$

In the first span, for values of $n = \frac{15}{16}, \frac{14}{16}, \frac{13}{16}$, and $\frac{12}{16}$, y_1 becomes 0.77 feet, 1.39 feet, 1.87 feet, and 2.22 feet. Drawing the several triangles for the weight on these successive joints, which are given at the bottom of the plate to a much enlarged scale, and whose vertices lie in a parabola, as indicated in § 30, we see that weights at the first and second joints will give positive moments at the first joint, and that these moments must, therefore, be added to the negative moment already obtained. The calculation of § 121 can be renewed here, if desired. Loads on five joints from the pier will have positive moments at the second joint; and their moments, added to the negative

moment at that joint due to complete load over the first and second spans, as shown by the ordinate from U, will make a resultant moment greater than the one for load on second span only: hence the former moment is the significant one.

Passing next to the two joints on the right of the first pier, we proceed in the same way. For loads on successive joints from the pier, or $n = \frac{1}{20}, \frac{2}{20}, \frac{3}{20}$, and $\frac{4}{20}$, $y_1 = 0.85$ feet, 1.56 feet, 2.11 feet, and 2.54 feet; while $y_2 = 0.30$ feet, 0.56 feet, 0.88 feet, and 1.21 feet. It will then be seen by the figure, or by calculation, that the removal of loads from the first two joints will increase the negative moment at the first joint, and that the removal of four loads to the right of pier B will increase the negative moment at the second joint of the second span. A load at the fifth joint will be found to give a negative moment at the second joint; and hence this investigation proceeds no further. All the moments are now obtained, and their amounts can be scaled.

It will be seen that the longest portion of the first span under positive bending moment at any time is about a hundred and sixty feet; and of the middle span, about a hundred and seventy feet. As the portion between two points of no bending moment acts, so far as chord-stresses are concerned, as if it were an independent span supported at these points, the height of the truss may be made twenty feet, nearly corresponding to one-eighth the span for a single-span truss. By varying the relative spans in any example, the distance in each span which shall be under positive moment can be brought to an equality, or nearly so; and hence a constant height for the whole girder may be chosen which shall give chord sections approximately alike in all spans. The negative moments at the middle of the centre span in this example are quite insignificant. Parts exposed to alternating stresses of opposite signs should be of heavier section than when called upon to resist compression only. As the ratio of H to the height of the truss is $2\frac{1}{2}$, the chord-stresses may be obtained from our diagram by multiplying the ordinates by this quantity; that is, changing the scale.

134. **Shear Diagram.** — The shear diagram can now be drawn by laying off the abutment and pier reactions as usual, equal to the segments of the load lines cut off by lines parallel to the closing lines, and connecting the extremities of the ordinates so obtained by the inclined lines seen in the figure. By a similar line of reasoning to that employed in constructing the diagrams for a two-span truss, the parabolas for maximum shear will be plotted upon the lines just referred to, and will answer to the combinations of load marked upon them. Some of the arrangements may not be probable ones; as, for instance, that a train shall cover A B, and another advance from C upon the middle span: but such combinations of parts of a train may occur as to render most of these positions possible in some localities. Enough shear curves are drawn to enable the stresses on all of the verticals and diagonals to be scaled, as the remaining web members will be put in by symmetry. The middle span will be symmetrical: the counters in the end spans will be found to move towards the abutments.

It has been stated that the shear parabola for a continuous girder as constructed on the two inclined lines gave, over the middle portion of the span, a stress a little in excess of the actual stress as obtained by separate constructions for the several partial loads. By drawing one moment diagram for each span for a rolling load covering one-half the span, and then plotting the special shear diagram for that case, the error of the middle ordinate can be found, and the whole error practically corrected by moving the curve slightly towards the tangents. The ends of the parabolas in the middle of each first panel are right. One such construction is carried out in the figure, for the middle span, as shown by the dotted moment polygon; and the slight error of the usual curve is seen, amounting to about a ton and a half at the maximum point, — a comparatively insignificant quantity.

135. **Extent of Chord subject to One Stress.** — To determine how far the stress in a chord will extend when a point of contra-flexure occurs in a particular panel, and there are two

diagonals in that panel, we must see which diagonal will be under stress at that time. Thus, in the end span, the point of contra-flexure reaches its extreme limit, T, when the middle span alone is completely covered with travelling load. The shear diagram for the end span will then be given by *ef*, and all the ties which slant down to the left will be in action: hence, taking moments at the joint to the left of T, we have tension in the seventh panel of the bottom chord from A, and compression thence to B, and compression in the seventh panel of the top chord, counting from the angle, with tension to the right of it.

136. **Deflection of a Continuous Girder.** — From the investigations of Chap. VI., it will be apparent that the deflection of a continuous girder under a load may be obtained by the method of area moments, as in more simple cases. It will be necessary to find the point where the tangent to the curve of the beam or truss is horizontal by dividing the moment area into two portions by an ordinate so placed that the area moment on one side, taken about its abutment, shall equal the moment of the area on the other side about that abutment. Either of these area moments divided by **E I** will give the maximum deflection. By making use of the original parabola and the negative area whose end ordinates are y_1 and y_2, the desired quantities are easily obtained.

NOTE TO § 131.—Areas may be easily calculated by the formula, Area = $\frac{2}{3}\,k\,l\left(1 - \frac{1}{N^2}\right)$; where N = No. of panels in span, and k = middle ordinate.

CHAPTER IX.

PARTIALLY-CONTINUOUS TRUSS.

137. **Settlement of Point of Support.** — The investigation has been founded thus far upon the assumption that the piers remain at the same elevations at which they were when the spans were joined on the false works, or what would be understood in an analytical investigation as a horizontal line. The trusses would then, if without weight, be without any strain. It is possible that one or more of the piers may settle a little, sooner or later, and the moments lately found will be disturbed by such a change. As the pressures on the points of support will be unlike, there may be more *compression* of one pier than of another. It is well to see how much the stresses may be altered by the compression of the foundation, or the settlement of a pier.

Let it be assumed that the bearing on the first pier B, Fig. 57, is lowered *one-fourth of an inch.* As the formula of § 130, which takes account of difference of level in supports, has a term involving **E I**, it will be necessary to find the value of these quantities. From the results previously ascertained, the average section of either chord in Fig. 57 is taken at 40 square inches. The depth of the truss from centre to centre of chords is 20 feet = 240 inches. The value of **I** will then be practically $2 \times 40 \times 120^2 = 1{,}152{,}000$. **E** may be taken as 26,000,000. As the units of **E** and **I** are pounds and inches, the areas **A**, **B**, and **C**, which have heretofore been expressed in square feet, may be reduced to square inches by multiplying by 144. When the tangent is drawn at B, A and C will be above

the horizontal line through B; when the tangent is drawn at C, B will be below C; while D is on a level with C. H = 50 tons = 100,000 lbs. The two equations of condition of § 131, modified by the formula of § 130, and divided by H, become

$$864\, y_1 + 240\, y_2 = 3\,(\text{A} + \text{B}) - \frac{6\,\text{E I}}{\text{H}}\left(\frac{\frac{1}{4}}{192 \times 12} + \frac{\frac{1}{4}}{240 \times 12}\right)$$

$$240\, y_1 + 864\, y_2 = 3\,(\text{B} + \text{C}) + \frac{6\,\text{E I}}{\text{H}} \cdot \frac{\frac{1}{4}}{240 \times 12}.$$

In order to retain the original areas, and to obtain y_1 and y_2 in feet as before, it will only be necessary to divide the last term of each equation by 144 to make them commensurable with the others. The last term of the first equation is, therefore,

$$\frac{6 \times 26{,}000{,}000 \times 1{,}152{,}000}{100{,}000 \times 144}\left(\frac{1}{9{,}216} + \frac{1}{11{,}520}\right) = 2{,}437.5.$$

The last term of the second equation becomes

$$\frac{6 \times 26{,}000{,}000 \times 1{,}152{,}000}{100{,}000 \times 144} \times \frac{1}{11{,}520} = 1{,}083.3.$$

Hence we obtain, by addition and subtraction,

$$\tfrac{1}{2}\,(y_1 + y_2) = \frac{1}{736}\,(\text{A} + 2\,\text{B} + \text{C}) - \left(\frac{1{,}354.2}{2{,}208} = 0.613\right)$$

$$\tfrac{1}{2}\,(y_1 - y_2) = \frac{1}{416}\,(\text{A} - \text{C}) - \left(\frac{3{,}520.8}{1{,}248} = 2.821\right).$$

It will now be seen that y_1 is diminished 3.43 feet, and y_2 is increased 2.21 feet, in every case, from the values previously obtained by a settlement at B of one-fourth of an inch.

How seriously the chord-stresses are changed by this small displacement of the pier-bearing is easily seen. A few of the changes are shown in Fig. 57 by the dotted closing lines, marked with accented letters. The points of contra-flexure are moved considerably also. As the height of the truss is twenty feet, the chord-stresses are increased or diminished two tons and a half for every alteration of one foot in the vertical ordinate below the particular joint. The supporting forces being altered, the shears, and hence the web-stresses, are somewhat changed also. If there is no reason for expecting a

settlement of one pier more than another, the possibility of a movement of either one should be considered. On account of the influence of change of level of piers upon the pier moments and reactions of a continuous girder, such a system seems best adapted to plate girders or heavy lattice riveted girders of moderate span.

138. **Partially-Continuous Girders.** — The evil effects of settlement may be avoided in two ways, — by the use of what we call partially-continuous girders, or by fixing the points of contra-flexure by *hinges*. As to the first method, —

Let the several spans of a bridge be erected independently, and swung clear of the false works; then let the upper and lower chords of successive spans be connected over the piers, the former by bolts, and the latter by compression-blocks or keys. The spans, when free from rolling load, will be subject to bending moments as single spans, none existing at points of support; and, when further loaded, the moments from rolling load only, computed as for a continuous girder, will be modified by the moments from steady load previously existing. If such a method is applied to the example of three spans just treated, the results in the two cases of continuous and partially-continuous girders can be readily compared. It is evident that the equations for pier moments must be applied to rolling-load polygons alone, the steady load being excluded entirely; and, as the *areas* for these spans have been carefully computed, it will be convenient to use $\mathbf{A} - \mathbf{A}'$, $\mathbf{B} - \mathbf{B}'$, and $\mathbf{C} - \mathbf{C}'$, in place of **A**, **B**, and **C**, for the areas due to rolling load.

But, as this example has been already worked out, if we note that the pier moments for steady load have been taken away by our device (which moments, by Case 2, § 131, are 35.39 feet), we may subtract this distance from all the values of y_1 and y_2 of § 131, and find the new quantities. Fig. 58 shows the results: the range of the points of contra-flexure is indicated, and the maximum ordinates are shown. The portions of either chord subjected to alternating stresses are very much reduced: the point of contra-flexure shifts but three panels from the

piers in the middle span, and five panels in the end spans. It is apparent that the positive moments near the middle of each span will be increased by the taking away of the pier moment due to steady load, and that the pier moments will all be diminished by this amount. The shear diagram for the centre span is unchanged; but the shear in the end spans is somewhat altered. The maximum *negative* moments must be investigated in the light of § 133.

There is one point to be especially noticed in finding the pier ordinates. The same letters refer to corresponding lines of Figs. 57 and 58. In subtracting the ordinate B E or C O of Fig. 57 from the remaining pier ordinates, the point N will be carried above C, as shown by the dotted line G N of Fig. 58. As this position indicates the existence of a positive moment at C, we must inquire whether the chords at that pier are adapted to resist that moment: if, as is probable, they are not, the ordinate at the pier B must be calculated anew for a girder of two spans, A B loaded and B C unloaded, remembering that the *travelling load alone* produces the moment at B. Thus was obtained B G, giving the closing lines A G and G C, drawn in full lines. By adapting the lengths of span to the intensity of travelling load it is possible to make the maximum positive and negative moments equal, or in any desired ratio.

Thus far the effect of settlement of piers has not been eliminated, and the arrangement here suggested would offer no advantages. If, however, it is suspected at any time that there has been a settlement of any one of the bearings, or to provide against such an occurrence, it is only necessary to loosen the connections over the piers when the bridge is empty, and to bring those pieces again just to a bearing or junction: the bridge is then restored to the condition under which it was designed. Whether it is practicable to apply such an expedient depends upon the supervision the completed structure will receive.

139. **Fixing Points of Contra-flexure by Hinges.** — If certain points in a girder extending across several spans are

hinged, so that no bending moment can exist at such points, the closing lines of the moment polygons must always pass through the hinges; and the stresses are at once made definite, whether the points of support settle or not, provided that enough hinges are supplied to locate the closing lines. A notable example of the introduction of this expedient occurs in the case of the Kentucky-river Bridge on the Cincinnati Southern Railway, — a bridge of three spans, of 375 feet each, shown in Fig. 59. As the gorge which this structure crosses is 275 feet deep, the ordinary false works for erection were out of the question; and the spans were built out from each cliff as projecting trusses, anchored back to the rock. The details of construction are not in place here; but suffice it to say, that, while the bridge was necessarily a continuous girder during erection, the fact that the two piers were built of iron, and might rise and fall some two inches from variations of temperature, while the natural rock abutments were unchangeable in level, required some expedient to obviate the great changes in stress which would otherwise occur. After the structure was completed, the connections in the lower chord at E and G were severed; and since the ties are concentrated at D, E, F, and G, the trusses were thus hinged on the upper chord-pins at D and F. A D is, therefore, an independent truss of 300 feet, supported at D by the truss D F, which overhangs its span B C 75 feet at each end. The moments and shears in the middle span will be influenced by a load on an end span: but the moments and shears in the end spans will depend entirely upon the load on those spans; that is, the stresses in A D will be those of an independent girder. E B is sometimes known as a cantilever.

The shear diagram for this bridge is not drawn; but the moment polygons are shown, — one for steady load, and the other for a complete live and steady load. As D′, E′, F′, and G′ are known points, the possible combinations of closing lines are at once drawn, and the points of contra-flexure shown by small circles determined. Under the supposition that trains

may at once cover any or all portions of the bridge, we shall have, for bridge empty, the closing lines A′ I L F′, with points of contra-flexure Q and R. For first span loaded we get A K L F′ when the entire middle span is under negative moments; for middle span alone loaded the points P and S are determined by A′ I L F′; the condition all spans loaded fixes V and W by A′ K N G′; and so on. If the bending moment at the middle of a single span of 375 feet for a travelling load which equals the steady load (an equality which is practically true for this bridge) be called 100, the maximum positive moment at 150 feet from A is about 63, the negative moment B′ K is 80, and the moment at the middle of B C is 60. By reason of the length of the bridge, and the fact that but one train will be on the bridge at one time, it is probable that some of these combinations will not occur. Of course only the possible moments will be considered in working out a design.

140. **Element of Indetermination in Multiple Systems of Bracing.** — The amount of shear which is found at any section of a continuous girder differs from that which would exist at the same section were the particular span independent, by reason of the influence of weight in one span upon reactions at other points. As the amount of vertical force which is thus added to or subtracted from the reaction at any point may pass across a span which has two or more systems of bracing by a choice of paths, the distribution of this portion between the different systems is indeterminate. Thus, in Fig. 57, the imposition of a complete load on the middle span, after the first and third spans are loaded, increases the positive shear in the third span by the constant amount $g\ h$, and diminishes the reaction at D by that amount. If there had been two systems of bracing, how much of this shear passed through either system would be unknown. The amount thus in doubt is not of serious consequence, the uncertainty being guarded against by a slight increase of section. A truss in which the stresses are perfectly determinate, exactly as when there are no superfluous pieces, is, however, the most satisfactory.

141. **Weighing the Reactions.** — It has been suggested that the girder may be designed for certain specific supporting forces, — either those which theoretically exist at the piers for a given load, or such other reactions as may make the maximum chord and web stresses what are desired; and then, upon putting the truss in position, that these reactions shall be weighed off to the proper amount by a lever or system of levers, and the bearing points shall then be blocked up until the index of the weighing machine returns to zero. For such a method of designing it will simply be necessary to lay off the reactions on the load line, draw lines from the points of division to the pole, and make the respective closing lines of the moment diagram parallel to the lines just drawn.

142. **Conclusion.** — The investigations which have preceded have not been conducted with a view to decide what type of bridge is the most economical, but to give the applications of this graphical method of analysis to such a variety of types, that the reader may be able, without difficulty, to find the stresses in any bridge-truss graphically. The advocates of the greater economy of continuous girders over single-span trusses rest their belief largely upon the reduction in absolute magnitude of the chord-stresses. The web-stresses are considerably increased. The weight of the structure can only be arrived at after the cross-sections of the pieces have been worked out, properly adapted to the stresses to which they are exposed. From the fact that a piece subject to reversal of stress requires, for equal safety, a larger cross-section than a piece exposed to but one kind of stress, or, in other words, a reduction of the safe intensity of stress per square inch, and that, on account of so large a portion of the two chords being liable to these alternating forces, special work must be put into the construction of such parts, a mere comparison of the numerical value of the stresses in two bridges, one continuous and the other discontinuous, will afford no good criterion of their relative cost. A fraction of a cent more per pound in the cost of executing the iron-work of a truss may cause it to be less economical in

first cost than a heavier bridge of a more simple design. Plate girders of moderate span on very stable supports, and with a comparatively large steady load, may well be built as continuous girders, for the full strength of the flanges and web may then be better developed; but, for skeleton trusses as usually built in America, we do not think the principle of perfect continuity over the piers is well adapted.[1]

The investigation of continuous girders is especially useful as introductory to the following chapter.

[1] For a discussion of these points, see Van Nostrand's Engineering Magazine, vol. xv., July to December, 1876; also Properties of Continuous Girders, by Charles Bender, New York, Van Nostrand, 1876.

CHAPTER X.

PIVOT OR DRAW SPANS.

143. **Draw-Spans.** — Following the treatment of the fixed spans of a bridge, either continuous or discontinuous over the piers, naturally comes the discussion of what is called the *draw-span*. While there are several simple ways of opening a small portion of a bridge to permit the passage of vessels, the only type of draw which requires special investigation is the one most commonly built at the present time, often of very large proportions, in which the span is capable of being revolved horizontally on a pivot or a wheel-ring at its middle, so as to open two channels, one on either side of the centre pier.

If the two parallel trusses of the draw-span are carried on the pivot by one cross-bearer, so as to be supported at a single point at the pier, the draw, when open, acts as two beams or trusses, each fixed and horizontal at one end, with a uniform load, arising from its own weight, over its whole extent. The two portions are usually, though not necessarily, equal in length, and they balance on the pivot; while the stability is assured by the wheels of a wheel-circle, which arrest any tendency to tip or cant. A certain amount of play permits the whole weight to be carried on the pivot. When the draw is closed, it may be elevated at the extremities by cams or wedges, so as to bring a greater or less pressure upon the piers; or it may simply swing into place over its wall-plates or seats, without practically pressing upon the piers until a travelling load comes upon it; or, finally, it may be so secured by horizontal locking-bolts, that, while these bolts do not bring the ends of the draw any more

closely in contact with the wall-plates, they prevent one end from rising from its seat when the travelling load first comes upon the other end.

144. **Draw as Two Single Spans.** — As the truss directly supported on the pivot is the simplest design for treatment, we will begin by referring to several modifications of this general type.

If the ends of the draw-span when it is closed are raised by mechanical means to such a height that the top chord over the centre pier is entirely relieved from tension, the two halves of the draw will then be independent spans. By making the top chord link with an elongated pin-hole, one can readily ascertain when no stress is transmitted through the member. So long as there is sufficient play or slack in the piece in question, the travelling load will also be carried by the two halves of the draw as if they were separate spans: hence the bending moments and shears for the closed draw will be ascertained as in Chap. II.; while the stresses caused by steady load in the open draw will be the same as in other cases, which will be treated more in detail presently. If the bending moment over the pier due to the steady load was just removed, and the outer ends of the draw elevated no farther, the travelling load would cause bending moments over the centre pier, and the draw would, when closed, come into the class of a partially-continuous girder of two equal spans, discussed in § 138.

145. **Draw as a Two-Span Continuous Girder.** — If the pivot-span is, on the other hand, when closed, supported at its ends at such an elevation that the moment over the centre pier is the same as exists in a continuous girder of two equal spans (in which case it is necessary, as seen by §§ 101, 102, that the supporting force supplied at each end by cams, hydraulic jacks, or otherwise, shall be, when there is no travelling load on the draw, *three-eighths* of the weight of *one span*), the draw when closed is circumstanced precisely like a continuous girder of two equal spans. If, then, we draw our diagrams for two equal, continuous spans, as described before, we have only to add the

moment polygon and diagram for shearing force for the draw open. A consideration of the method given for a beam overhanging at one end will show the construction in this case. Here the beam overhangs at both ends, and the two supports are united in one, the beam balancing upon it.

Construct, if it has not already been done, the equilibrium polygon for the truss loaded with its own weight. Remember that the *end joints* each carry one-half of a panel weight: therefore, from the points where the polygon cuts the verticals from the extreme ends of the span, draw parallels to lines in the stress diagram from 0 to the *extreme* ends of the load line, and these lines must cut the vertical under the centre pier in the same point. By referring to Fig. 60 we can see that the draw, when open, is in the condition of the beam A B C, if the abutments A and C are removed. The equilibrium polygon will be A′ E′ C′, and A′ B′ and B′ C′ would be tangents to the parabola drawn through the vertices of the polygon. The ordinates to the polygon thus intercepted will be, when multiplied by H, the bending moments at the joints for the open draw.

For the shear diagram draw lines from the extremities *a* and *c* of base line for shearing force to points *b* and *f*, below and above the base line a distance equal to the weight of one span, or one-half of the load line, *b f* being the reaction of the centre pier, and equal to the entire weight of the unloaded draw. The ordinates at the middle of the panels will show the shearing force on the left of a section when the draw is open.

The diagrams for this case are thus completed; that is, these lines just described are to be added to the diagrams required for two equal continuous spans. When the draw is open, there will be tension throughout the upper chord, and compression throughout the lower chord, and braces will all slope in one direction from the centre pier to either end. The stresses when the draw is closed, and the direction of the braces required, will be as stated in Chap. VII. If the supporting forces at the ends exceed or fall short of the amount before stated, three-eighths of the weight of one span, the case

requires a different treatment. From the power applied to the cam, or jack, the actual amount of reaction may be approximated to, and the new state of affairs readily be classed under a case soon to be noticed.

146. **Draw balanced on Pivot when closed.** — More commonly a draw is screwed up at the centre by shortening the top chord link, or raising the pivot, until almost or quite the entire weight and bearing is upon the pivot, and the ends of the truss scarcely do more than touch the bearings on which they close, as is shown by the immediate tilting up of one end of a draw when a train enters the other end. The draw may then practically be considered as poised on the centre when shut and unloaded, with the same stresses as when open. Let A B C, Fig. 60, represent the draw closed, balanced on the pivot B, and barely in contact with the abutments at A and C. The curvature in the figure is exaggerated; but every draw is theoretically curved in this way, when supported in the middle, and deflected under its own weight, even when it is actually straight and horizontal, as is proved by the existing tension in the top chord, and compression in the bottom chord; and the points A and C can be brought to a level with B only by a *reversed camber* previously produced in the truss.

147. **Action of a Rolling Load.** — When a rolling load comes on at D, in the sketch below A C, the *farther* end F rises, the truss is carried by two supports, with one end overhanging, and the construction of the diagrams will follow the method described in § 97. As recently stated in § 145, A′ E′ C′ B′ is the diagram for bending moment of A B C, the draw open, and also for the draw closed, but not raised at the ends by cams. By drawing T′ D′ as the new portion of the polygon required by the load T D, we have D′ T′ E′ C′ as the new equilibrium polygon. Any load on D E will not alter the bending moments in E F: hence C′ B′ is still one of the closing lines, and B′ D′ must be the other required to complete this diagram. The point of contra-flexure is near T′. A line in the stress diagram drawn parallel to B′ D′ will give us the reaction at D, equal to *a d;*

and the lines $d\,t\,e$ with $f\,c$ will determine the ordinates for shearing force.

The farther end of the draw will rise still higher as the load advances over the first span, and finally reaches the centre pier. The progress of the load beyond this point, while it at the same time covers the first span, will cause F to move down again; but the bridge will not touch the bearing on G until the train or other moving load has advanced a certain, often a considerable, distance over the free span, — sometimes more than one-fourth of the space from the centre pier. As, when the draw touches three points, another method of analysis must be applied, the important position to be determined is that of S, the front of the rolling load on I K L, when the end L of the beam is just forced down to its abutment.

148. **Condition that Draw shall rest on Three Points.** — Draw through K a tangent Q K N to the beam at that point. When the beam rests on the abutments, the vertical deflections at the ends from the tangent at K are N L and Q I. The points of support I and L are below the horizontal line R K O a distance R I or O L. If we add I R to Q I, and subtract its equal O L from N L, we have a similar proportion to the one deduced for a two-span continuous truss; namely, —

Deflection N O : deflection Q R = K O : K R = 1, or N O = Q R.

Now N L is proportional to the *area moment* of the span K L, as explained in § 104; Q I is proportional to the area moment of the span K I; and O L or I R is proportional to the original area moment of either span before any rolling load has come upon it, or to the area C′ B′ E′ multiplied by the distance of its centre of gravity horizontally from the vertical through L. Call this area moment **C c**. The polygon I′ E′ L′ being drawn, the condition that the end L shall rest upon the abutment is therefore satisfied when the area moment to the left of E′ K′ plus **C c** equals the area moment to the right of E′ K′ minus **C c**. This equation will determine the position of K′, as in former cases.

To find the distance $E' K'$, or y_o, for any position of the head of the rolling load between S and L: Draw a straight line from I' to E', and one from E' to L'; call the respective areas between these lines and the moment polygon **A** and **B**, **A** being the area belonging to the side which is entirely loaded; and the respective distances of their centres of gravity horizontally from I' and L', **a** and **b**; then by the same course of reasoning given in § 106, and from the relation stated above, we have, letting l equal either span,

$$\mathbf{A\,a} - \tfrac{1}{3}\, y_o\, l^2 + \mathbf{C\,c} = \tfrac{1}{3}\, y_o\, l^2 - \mathbf{B\,b} - \mathbf{C\,c},$$

$$y_o = \frac{3}{2\,l^2}\,(\mathbf{A\,a} + \mathbf{B\,b} + 2\,\mathbf{C\,c}).$$

In the special case now under discussion it must be noted, that when we find a point K', satisfying the above condition, so placed that a line parallel to the last radiating line in the stress diagram to the *extremity* of the line of loads cuts $E' K'$ above K', the span has risen from the abutment; for it is necessary that a parallel to $K' L'$ shall cut off some portion of the load line to give any supporting pressure on L. When one end of the span is off the abutment, the head of the load being to the left of S, see § 147. If the polygons are drawn as usual in these pages, $I' E'$ being common to all of the polygons for movements of the head of the load from S towards L, we have **A a** and 2 **C c** constant, as well as the factor outside of the parenthesis; so that for each new position of the load we have only to calculate the quantity $\frac{3\,\mathbf{B\,b}}{2\,l^2}$, and add to the previous constant quantity, to obtain $y_o = E' K'$.

149. **Shear Diagram, and Points of Contra-flexure.** — The shear is then readily obtained. A load from I to S gives the diagram *i n l s c*: when the load extends to L we have the symmetrical curve $I' E' V'$ and the shear diagram *a u w k v c*. The increase of rolling load from S to L diminishes the supporting force at I from *a i* to *a u*, and manifestly the supporting force at I will be a maximum when the span I K alone is cov-

ered by the travelling load. Parabolas drawn on the limiting lines of the shear diagram will give the maximum ordinates. An example will be given in detail soon.

The points of contra-flexure will be found nearer the ends for given loads than in a continuous girder of two spans, as might be expected, since the load has first to overcome or neutralize the initial negative moments of flexure: consequently the bending moment over the centre pier is greater than in the case of a continuous girder. Most trusses for draws have a greater depth at the centre pier than at the ends, thus diminishing the chord-stresses near that point below the amounts which would exist in a truss with parallel chords and with an average height of this draw. If the variation of height is expected, by its effect on the moment of inertia at successive cross-sections, to seriously influence the deflection, the ordinates which make up the moment areas must be changed in the ratio of the change of **I** before the area moments are computed for the preceding equation for y_o. In ordinary cases, the assumption of a constant moment of inertia will lead to no serious error.

150. **Draw with Locked Ends.** — When the ends of the draw are prevented by locking bolts from rising from the abutments, the action will be similar to that produced by hanging an additional but variable weight at F, just sufficient to bring F in contact with G. The greatest force will be required at F when the rolling load extends from D to E. It is readily seen, that, if F must always remain on G, the condition required for the previous case, § 148, instead of being limited to loads which give a pressure on L, must be satisfied for every position of the load from D to F; that is, we must always have

$$y_o = \frac{3}{2\,l^2}(\mathbf{A}\,\mathbf{a} + \mathbf{B}\,\mathbf{b} + 2\,\mathbf{C}\,\mathbf{c}),$$

thus determining K′. If, then, the line to be drawn in the stress diagram, parallel to K′ L′, passes beyond the end of the load line, the additional length of load line required to meet it will be the upward pull on the bolt at L; and this pull will

increase, commencing with zero for the head of the load at S, until the rolling load retires to K, and will then diminish to zero again when the load entirely moves off at I. This modification makes the only difference in the treatment of these two cases. In the shearing diagram, the amount of stress on the locking bolt, being opposed in direction to a supporting pressure, is to be laid off at *c* upwards or at *a* downwards, and from its extremity will then be drawn a line parallel to and taking the place of *cf* or *ab*. This pull affects the stresses in all portions of the draw, and shifts the points of contra-flexure.

151. **Draw with Ends partially lifted.** — Suppose that the draw, in place of being circumstanced as in § 146, is raised at the ends by cams, or jacks, but that the supporting forces do not equal those required by § 145, where A and C are on the same level with B. Find, from the known power applied to the cams, the amount of supporting force at each end when the draw is unloaded; lay off these amounts, each on the proper end of the load line, and draw lines from the two points of division so obtained to the pole of the stress diagram, usually marked 0; construct the moment polygon for the unloaded spans; and then draw lines from the extremities of the polygon to the centre vertical parallel to the lines just drawn in the stress diagram. The ordinates so cut off will be the ones required to determine the bending moments on the closed and unloaded draw, and the area moment on one side (or the moments on each side, if not the same, owing to different supporting forces at the ends) will represent the quantity to be used, instead of **C c**, in the equation for y_o.

The ends of the draw may be raised so as to give a pressure of more than three-eighths the weight of one span on each abutment. It will then be necessary to determine the amount, and proceed by the steps just described.

152. **Remarks on the Preceding Cases.** — It is evident that the quantity **C c** is proportional to the difference of level between the centre pier and the end pier, and hence is related to the quantity v of the formulæ in §§ 91, 130. From the

peculiarity of the draw-span, that it may swing over the abutments without pressing upon them when unloaded, this area moment is readily ascertained; and, being of the same form as **A a**, &c., it makes the equation very simple. A general case might be made of the difference of level just referred to. If A and C are above B, we shall have the last case of § 151. If A and C are sufficiently above B, we shall have no pressure on B, and hence one single span, unless a tension can be exerted at B. If A and C are level with B, **C c** will equal zero, and we have the case of § 145. If A and C are below B, we have the cases of §§ 146 and 151. An equation formed from that of § 106, by introducing two terms in **C c**, one for each span, will apply to this general discussion, and will give us, when the spans are equal, the equation of § 148.

153. **Example.** — To illustrate the principles thus far laid down, let us take a draw-span represented in Fig. 61, and find the stresses on the different parts under the case of § 150. The draw is 240 feet long, making two spans of 120 feet each ($= l$) divided into twelve panels of 20 feet each. The height at the centre is 25 feet, and at the ends 20 feet. The dead weight is assumed at 10 tons per panel of one truss, and the live load is also 10 tons per panel. Simple data are taken for brevity, and to keep a small figure distinct. To have the moment curves well separated, let H = 60 tons. The load line 1–2 is 240 tons, the maximum load; and 0 is opposite its middle. A and C carry 10 tons each, the other joints 20 tons each, when loaded. A′ M C′ is the polygon for the fully-loaded draw. Through M draw A″ M C″ for the draw unloaded: the side of the first polygon at M, which is parallel to 0–7, being common to both, construct this polygon each way from M, with ten tons per joint, terminating at C″ and A″ with lines parallel to 0–8 and 0–9. As five tons, when the draw is free or open, are carried at A and C, draw from A″ and C″ the lines A″ B″ and C″ B″ parallel to 0–5 and 0–6. The ordinates between A″ M C″ and A″ B″ C″ are proportional to the bending moments at different points of the draw open, or closed and unloaded.

Calculate the area $A'' M B'' = \mathbf{C}$; find the distance of its centre of gravity from the vertical through A, and denote it by $\mathbf{c}$. (See § 105. As the angles of $A'' M$ lie in a parabola, $\mathbf{c}$ may be taken as $\frac{3}{4}$ A B. By subtracting a parabolic segment from a triangle, the centre of gravity of a parabolic spandrel is easily obtained, as above. The small portions by which the polygon differs from the curve balance about the centre of gravity of the segment; and hence, if the area $\mathbf{A}$, bounded by the polygon and the straight line connecting its extremities, is computed, the exact position of the centre of gravity of the area $\mathbf{C}$, found by subtracting this first area from the triangle, is given.)

Draw a straight line from A' to M, and find $\mathbf{A}$, the included area between it and the polygon $A' M$; $\mathbf{a} = \frac{1}{2}$ A B is the distance of its centre of gravity from A: then, since $\mathbf{B}\,\mathbf{b} = \mathbf{A}\,\mathbf{a}$ for a full load, the formula of § 150 becomes simply

$$y_0 = \frac{3}{l}\left(\tfrac{1}{2}\,\mathbf{A} + \tfrac{3}{4}\,\mathbf{C}\right) = \frac{3}{4\,l}\left(2\,\mathbf{A} + 3\,\mathbf{C}\right).$$

Lay this value off at $M B'$; draw $A' B'$ and $B' C'$, and thus determine the bending moments for the draw closed and loaded. Draw a straight line from C'' to M; compute $\mathbf{B}$ for this side; and, using $\mathbf{A}$ as before, find

$$y_0 = \frac{3}{4\,l}\left(\mathbf{A} + \mathbf{B} + 3\,\mathbf{C}\right) = M B'''.$$

$A' B'''$ and $C'' B'''$, as well as $A'' B'''$ and $C' B'''$, will give the bending moments when one span is loaded, and the other unloaded.

Draw from 0 lines parallel to those which meet at B', B'', and B''', thus finding the supporting forces, and plot them at a, b, and c. These reactions will be $a\,e$, $f\,o$, and $d\,c$, when both spans are loaded; $a\,k$, $i\,x$, and $n\,c$, when A B is loaded, and B C unloaded; $a\,l$, $m\,s$, and $g\,c$, when the load is upon B C alone. The maximum pressure at A must be $a\,k$, and the maximum shear on the locking bolt or tension on the abutment is $a\,l$. The lines $a\,h$ and $l\,m$ are inclined at 10 tons per panel run, and $e\,f$ and $k\,i$ at 20 tons per panel. The distance $a\,l$ must equal $e\,k$ by

§ 120. Parabolas drawn on these tangents as usual, beginning in the middle of the first panel from each point of support, will complete all *necessary* lines in the shear diagram. As the two halves of the draw are alike, the diagram for one-half is sufficient. To construct these parabolas, see § 118.

154. **Discussion.** — If the draw had parallel chords, the diagrams would now be completed; but, owing to the changes of depth, other moment polygons are needed. Those on the left span are drawn for loads extending from A to the joint whose letter stands at their left extremity; those on the right, for loads extending from B to the joint whose letter is placed at their right extremity: by combining these polygons with a full load polygon, or an unloaded polygon on the other side, or with one another, any variation of loading is obtained. As a load on B does not alter the bending moment, the polygon for a load which includes P becomes the one from A′. Quite a number of pier ordinates have been plotted, after calculating **A** and **B** as usual; and enough are given here to show the movement of the lines. The shears for partial loads can thus be obtained directly, if desired; and the maximum moments can be scaled. The statement affixed to the shear parabolas indicates the loaded portion for the maximum shear in each panel.

As the load advances from A to B, then, the span B C being fully loaded, we find that the ordinates for the maximum shears in the successive panels occur immediately in front of the load, and terminate in the points u, v, w, &c., of a parabola described on the lines $l\,p$ and $p\,f$, and ending in the middle of the first and last panels. If B C were *unloaded*, these ordinates would terminate in a parabola drawn on $a\,q$ and $q\,i$; but, as the former parabola includes the latter, the former alone is wanted. As a load on B A alone will cause a greater reaction at A than when B C also is loaded, therefore the increments of load from B towards A, while B C is unloaded, will give us points on a parabola drawn on $k\,q$ and $q\,h$. Similar curves might be drawn on $t\,r$ and $r\,g$, $o\,r$ and $r\,n$.

The point of contra-flexure, advancing from the outer end as

a load enters upon the draw, will be seen to move no nearer the centre pier than the third panel from the end: consequently K R will always be in tension, G B will always be compressed, while D K and A G must be designed to withstand either stress. The maximum ordinate at any joint being readily selected from the figure A′ M C′ C″ B′ A″, and multiplied by H, if we divide by the height of the truss at that joint we shall have the *horizontal* stress on that side of the joint not touched by the diagonal in action at the time. The horizontal component in the inclined chord must be increased in the ratio of the actual length of that portion of the chord to the length of a panel horizontally, in order to obtain the direct stress. A curved piece may be treated as straight between the two joints for finding the direct stress: its curvature introduces a separate bending moment on the piece itself.

155. **Web-Stresses.** — Finally, to find the stresses in the diagonals and verticals, compare § 63, &c. Take, for example, the pieces L Q and P Q. The maximum shear in the panel L P will be the ordinate from *a b* to *w:* lay off this ordinate from *w* to *o* in the lowest figure. This shear will be caused by a rolling load from A to L inclusive, together with one from B to C: as it is negative, it is plotted downwards. The moment polygon is L′ M C′, and the ordinates for bending moment under that load at L and P will be L″ N′ and P′ Q′. Multiply these ordinates by H, and divide by L N and P Q respectively; thus obtaining the *horizontal* stress towards N and the stress in P L. Lay off *w* L equal to the latter, and, drawing *o* Q parallel to N Q, make the horizontal distance of Q from *o* equal to the former. Negative shear and negative bending moments turn this diagram around, and bring the top chord-stress at the bottom of the diagram, or the reverse of Fig. 31. If no error has been made, Q L, when drawn, will be parallel to Q L of the truss, and will give the amount of existing tension; while the line marked P Q, drawn vertically from L, will be the stress in the vertical P Q. The remainder of the figure applies to the other pieces of the web, as shown by the letters. B R will

undergo double the compression shown in the diagram, as it resists the action of the inclined pieces in both halves of the draw.

156. **Remarks.** — A study of the diagrams of Fig. 30, and the explanations therewith, will show what modifications would be introduced in Fig. 61 by the inclination of the bottom chord, the substitution of struts for ties, or of a load on the top chord for one on the bottom chord. On account of the existing shears and moments, the inclination of the chord, opposed to that of Fig. 30, is favorable to the ties. In some examples with more numerous panels, joints near the pier may be subjected to maximum negative moments by partial loads, as explained in § 111. If all the polygons are drawn as here, the maximum moments can be selected at sight.

157. **Changes by Omission of Bolt.** — Without drawing other diagrams, we can determine, by inspection of Fig. 61, what changes would be effected in the draw by the removal of the locking bolts, bringing it under the case of § 146. Both when the draw is entirely loaded, and when the rolling load is altogether removed from the draw, there is no force exerted on the bolts. The absence of the bolt will, therefore, make no change in the position of B′ and B″. The point B‴, found when one span only is fully loaded, will be situated a little nearer M, owing to the omission of the moment over B caused by the pull on the bolt multiplied by the span B C. The point of contra-flexure will, therefore, come somewhat nearer the centre pier; and, if it gets into the next panel, the extent of chord subject to but one kind of stress will be diminished. All the values of y_0 which would give any pull on the bolt, and which correspond to loads on any portion of R S, Fig. 60, will be slightly less.

As $l\,u\,v\,f$ is the curve for a load advancing from A while the other span is fully loaded, and as, under these circumstances, the holding-down bolts cannot be in action, this parabola will not be changed. When A B, and it alone, is covered with a rolling load, the supporting force at A is diminished by

the amount of the *pull* $n\,c$ on the abutment at the other end. Remove the bolt at C, and the point k will now be found to have moved farther from e by the amount $n\,c$ or $a\,l$: that is, to find k, lay off $l\,k$ from a. This increase of supporting force at A will affect the shear at all points of the span, when A B is loaded, by just this amount; and therefore $k\,i$ will move parallel to itself to the new distance from a, just stated. The parabola $k\,h$ will rise to its new lines, increasing the shear in panels near A.

158. **Draw of Three Spans.** — It is more common to design a draw-span with a middle panel, and to carry the trusses on two *transverse* beams or girders which are placed directly beneath the joints of this panel. In this construction the open draw is supported at two points equidistant from its middle; and, when closed and loaded, the draw may be carried by the abutments also, being thus divided into three spans, — two usually equal end spans, and one short middle span of the length of the middle panel. The transverse beams will rest at four equidistant points upon a deep circular girder, which, in turn, is either carried by the pivot, or rests directly upon the live ring of wheels below the girder. In some cases the attempt is made to distribute the weight between the ring and the pivot in some desired proportion. We propose to treat in this section the case where the draw rests directly upon the wheels.

We draw, as before, the moment polygon for the unloaded draw open, or closed, but not bearing upon the abutments. This construction is seen at A′ D′ T′ S′, Fig. 62. In any determination of the pier ordinates for a partial or complete load, it is necessary to know the area moment which indicates the difference of level between A and B. Let the span A B $= l_1 =$ C D; the span B C $= l_2$. The draw, when open, has a horizontal tangent at E, its middle point. If $\mathbf{C}\,\mathbf{c}$ is the area moment of A′ B′ S′, and $\mathbf{D}\,\mathbf{d}$ the area moment of B′ S′ U′ E′, both about A, we have the

	deflection of	A	below	E = A S	is proportional to	$\mathbf{C}\,\mathbf{c} + \mathbf{D}\,\mathbf{d}$,
	"	B	"	E = S T	" "	$\mathbf{D}\,(\mathbf{d} - l_1)$:
∴	"	A	"	B = T A	" "	$\mathbf{C}\,\mathbf{c} + \mathbf{D}\,l_1$.

159. **Values of Pier Ordinates.** — When a load comes upon the draw, and the ends are in contact with the abutments, either exerting a pressure there, or locked down, we shall have the condition of things represented by the lower figure. Upon drawing the tangents to the curve of the beam (L N at G, and O P at I), and noting that Q F = R K = T A, we may write two equations for y_1 and y_2, the ordinates at G′ and I′, referring to §§ 104 and 124 for the general form, and denoting by **A a** and **B b** the area moments of each span l_1 cut off by F′ G′ and K′ I′ as before: —

$$\frac{LQ}{NI} = \frac{LF + FQ}{NI} = \frac{\mathbf{A\,a} - \frac{1}{3} y_1 l_1^2 + \mathbf{C\,c} + \mathbf{D}\,l_1}{\frac{2\,y_1 + y_2}{6}\,l_2^2} = \frac{l_1}{l_2},$$

$$\frac{PR}{OG} = \frac{PK + KR}{OG} = \frac{\mathbf{B\,b} - \frac{1}{3} y_2 l_1^2 + \mathbf{C\,c} + \mathbf{D}\,l_1}{\frac{y_1 + 2\,y_2}{6}\,l_2^2} = \frac{l_1}{l_2}.$$

Clearing of fractions, transposing and factoring, we easily deduce

$$2\,(l_1 + l_2)\,y_1 + l_2\,y_2 = 6\left(\frac{\mathbf{A\,a}}{l_1} + \frac{\mathbf{C\,c}}{l_1} + \mathbf{D}\right), \qquad (1)$$

$$l_2\,y_1 + 2\,(l_1 + l_2)\,y_2 = 6\left(\frac{\mathbf{B\,b}}{l_1} + \frac{\mathbf{C\,c}}{l_1} + \mathbf{D}\right); \qquad (2)$$

which equations are very readily solved for any given case. As **a**, **b**, and **c** are fractions of l_1, the second members are simple. By adding and subtracting, if more convenient, we at once obtain the half-sum and half-difference of y_1 and y_2,

$$\tfrac{1}{2}\,(y_1 + y_2) = \frac{3}{2\,l_1 + 3\,l_2}\left(\frac{\mathbf{A\,a}}{l_1} + \frac{\mathbf{B\,b}}{l_1} + \frac{2\,\mathbf{C\,c}}{l_1} + 2\,\mathbf{D}\right), \qquad (3)$$

$$\tfrac{1}{2}\,(y_1 - y_2) = \frac{3}{2\,l_1 + l_2}\left(\frac{\mathbf{A\,a}}{l_1} - \frac{\mathbf{B\,b}}{l_1}\right). \qquad (4)$$

G′ O′ and I′ N′ being thus obtained, the lines F′ O′ and K′ N′ will determine the moments, the reactions, &c. The polygon for the unloaded draw has been added to this diagram.

160. **Special Treatment.** — In case the end of the span, for instance at K, is not locked down, it may rise when a load enters at F. This will be indicated by obtaining a tension in

place of a pressure for the reaction at K: then K′ N′ must be drawn parallel to the line from the extreme end of the load line to the pole, giving an independent value of y_2, which is then to be substituted in equation (1) above, and the value of y_1 at G thus obtained.

It may rarely happen in some designs, under a very heavy travelling load, that, when there is no locking bolt to prevent, a load on the span F G may raise the draw, not only from K, but also from I. This occurrence will reduce the draw to a truss of the span F G, with the portion G K overhanging, § 97. In that case the straight line K′ N′, referred to in the first paragraph of this section, will be continued to the vertical through G′; and any value of y_1, obtained as above, which is greater than the ordinate intercepted at G′ by this line, cannot exist, for tension would then be necessary at K.

Again: it is plain, that, if I′ N′ or y_2 is small enough, O′ N′ and N′ K′ may become one straight line when the lines which cut off the reaction at I from the load line coincide, giving a pressure of zero. It may be possible, therefore, when locking bolts are used, and sometimes when they are not employed, that a sufficient load on one span, as F G, may reduce the pressure at I to zero, lifting the draw and circular girder from the wheels on that side: hence if N′, as found by the above equations, falls nearer I′ than a straight line from O′ to K′ would locate it, tension would be necessary at I to keep the girder on the wheels. As this force cannot be supplied, the draw for that given distribution of load must be treated as a draw of two unequal spans, F G $= l_1$ and G K $= l_1 + l_2$. The equation of § 148 then becomes, upon substituting the proper quantities,

$$\frac{\mathbf{A}\,\mathbf{a} - \frac{1}{3} y_1 l_1^2 + \mathbf{C}\,\mathbf{c} + \mathbf{D}\, l_1}{\frac{1}{3} y_1 (l_1 + l_2)^2 - \mathbf{B}'\,\mathbf{b}' - \mathbf{C}\,\mathbf{c} - \mathbf{D}\, l_1} = \frac{l_1}{l_1 + l_2},$$

in which $\mathbf{B}'$ is the area cut off by a line from K′ to G′, and $\mathbf{b}'$ is the distance of its centre of gravity from K. From this equation is deduced

$$y_1 = \frac{3}{2\, l_1 + l_2} \left(\frac{\mathbf{A}\,\mathbf{a}}{l_1} + \frac{\mathbf{C}\,\mathbf{c}}{l_1} + \mathbf{D} + \frac{\mathbf{B}'\,\mathbf{b}'}{l_1 + l_2} + \frac{\mathbf{C}\,\mathbf{c}}{l_1 + l_2} + \frac{\mathbf{D}\, l_1}{l_1 + l_2} \right).$$

This ordinate is to be plotted at G′ O′, and lines drawn from O′ to F′ and K′. The preceding statements indicate when one and when another of these equations is applicable. The shear diagram can then be constructed, and the case completely solved. If the ends are raised by cams, &c., **C c** is to be found from the known reaction, as in § 151.

161. **Circular Girder carried by Pivot; Draw of Two Spans with Tipper.** — Where the weight on the circular girder is transferred directly to the pivot, the wheel-ring is placed below the circular girder as usual; but sufficient clearance is allowed between the lower flange of the girder and the wheels to prevent any weight from resting upon them when the draw is closed. When the draw has swung off from the abutments in opening, the wheels will check any extreme oscillation of the draw. So far as the truss itself is concerned, it is supported, when open, upon two points a centre panel length apart, as in the previous case; and, when closed, upon the abutments, and on two points, which, under an unsymmetrical load, change their elevations by the rocking or tipping of the circular girder on the pivot. These two central points are, consequently, not at a constant elevation above the abutments. If Fig. 63 represents this case, it will be seen that the only point which does not change its position on the supposition that the points B and C in contact with the beam, in shifting to K and L, move in a vertical line, is the point T, midway between and in the straight line joining B and C, or K and L. All deflections will be referred to this point. (The above supposition is the customary one in investigating flexure of beams, that the span is unchanged, or the curved line is the same in length as the original straight one.) In the beam A D, therefore, we propose to use the deflection of A from a horizontal line through B, that is, G A or F A — F G, and then to write the usual proportion for the beam I N, referred to a horizontal line P X through T.

162. **First Condition for Pier Ordinates.** — When the tipper B C, which rocks on E, is level, the draw being open, or closed

and unloaded, the distance G A will, by the notation of the previous section, be proportional to **C c** + **D** l_1 = deflection of abutment below B when B C is horizontal. These areas are marked in the figure, A′ F′ B′ = **C** and F′ G′ E′ B′ = **D**. In endeavoring to determine the two moments or pier ordinates at K and L for a partial or complete load, we must establish two conditions involving the two unknown quantities. One condition is especially simple. As the pivot or fulcrum of the lever K L is midway between K and L, it follows that the *reactions* at these two points must be equal to one another. If we suppose, for the present, that I′ K′ L′ N′ are the desired closing lines, let us prolong K′ L′ until it meets the verticals from I′ and N′ at Q′ and U′. Upon drawing lines 0–3, 0–4, and 0–5, in the stress diagram, parallel to I′ K′, K′ L′, and L′ N′, the two intercepted parts of the load line cut off by these three lines must be equal. If, then, the vertical sides of the two triangles just constructed in the stress diagram are equal, and the side 0–4 parallel to K′ L′ is common to both, the vertical sides of the similar triangles I′ Q′ K′ and L′ U′ N′, which have Q′ K′ = L′ U′, must be equal, or I′ Q′ = N′ U′: hence K′ L′ will always be drawn parallel to the original closing line I′ N′; and this condition must be satisfied for a draw of two equal spans. If the spans are unequal, it follows that I′ Q′ : N′ U′ = I K : L N. As this condition fixes the direction of K′ L′, it will only be necessary to find one ordinate to it at the most convenient point to completely solve the problem.

163. **Second Condition for Pier Ordinates.**—If I K L N represents the loaded draw, K and L the rocking supports carried by O, S the middle point of the *beam*, and T the point below it in the straight line K L, this point T is above I a distance P I, equal to G A, the original deflection of A below B, the measure of which was given in the last section. Draw the moment polygon I′ S′ N′; cut off the area **A** by the straight line I′ R′, and the area **B** by the straight line N′ V′. Let I′ K′ L′ N′ be the desired closing lines; draw I′ S′ and S′ N′; also draw the vertical S′ T′. Let area R′ I′ S′ = **K**; S′ V′ N′ =

N; R′ K′ T′ S′ = **E**; T′ L′ V′ S′ = **F**; and denote the distances of their respective centres of gravity horizontally from I′ and N′ by the usual small letters.

Drawing a tangent Q U to the bent beam at S, and a line R V parallel to it through T, we have, if P X is the horizontal line through T,

$$\text{P R} = \text{V X, or Q I} + \text{I P} + \text{Q R} = \text{U N} - \text{N X} - \text{U V}.$$

Now, I P = N X = G A has been proved proportional to $\mathbf{C\,c} + \mathbf{D}\,l_1$; Q R = U V, being the deflection of K below a tangent through S, is proportional to $\mathbf{E}\,(\mathbf{e} - l_1)$; Q I is measured by the area moment to the left of S′ T′, or $\mathbf{A\,a} + \mathbf{K\,k} - (\mathbf{K\,k} + \frac{1}{3}\,\text{R}'\,\text{K}'\,.\,l_1^2 + \mathbf{E\,e})$. A similar expression on the right of S′ T′ will measure U N. Substituting these values in the above equation, noticing that the areas **K** and **N** cancel out, and remembering to write moments on opposite sides of the tangent with opposite signs, we get

$$\mathbf{A\,a} - \tfrac{1}{3}\,\text{R}'\,\text{K}'\,.\,l_1^2 - \mathbf{E\,e} + \mathbf{C\,c} + \mathbf{D}\,l_1 + \mathbf{E}\,(\mathbf{e} - l_1) =$$
$$- \mathbf{B\,b} + \tfrac{1}{3}\,\text{V}'\,\text{L}'\,.\,l_1^2 + \mathbf{F\,f} - \mathbf{C\,c} - \mathbf{D}\,l_1 - \mathbf{F}\,(\mathbf{f} - l_1); \text{ or}$$
$$\tfrac{1}{3}\,(\text{R}'\,\text{K}' + \text{V}'\,\text{L}')\,l_1^2 + (\mathbf{E} + \mathbf{F})\,l_1 = \mathbf{A\,a} + \mathbf{B\,b} + 2\,\mathbf{C\,c} + 2\,\mathbf{D}\,l_1.$$

From the figure we see, that, since R′ V′ is a straight line, $\mathbf{E} + \mathbf{F} = \frac{1}{2}\ (\text{R}'\,\text{K}' + \text{V}'\,\text{L}')\ l_2 = \text{S}'\,\text{T}'\,.\,l_2 = y_o\,l_2$; so that the above equation may be written

$$\tfrac{2}{3}\,y_o\,l_1^2 + y_o\,l_1\,l_2 = \mathbf{A\,a} + \mathbf{B\,b} + 2\,\mathbf{C\,c} + 2\,\mathbf{D}\,l_1, \text{ or}$$
$$y_o = \frac{3}{2\,l_1 + 3\,l_2}\left(\frac{\mathbf{A\,a}}{l_1} + \frac{\mathbf{B\,b}}{l_1} + \frac{2\,\mathbf{C\,c}}{l_1} + 2\,\mathbf{D}\right),$$

which is identical with (3) § 159.

The construction with the tipper or rocking circular girder then becomes simply, Make S′ T′ equal to y_o, draw K′ L′ parallel to I′ N′, and complete the figure with I′ K′ and L′ N′. The shear diagram for this load is seen below.

164. **Special Treatment.** — When the end of the draw is not locked down, all loads on one half of the draw, and even a portion extending upon the other half, will cause the unloaded end to rise from the abutment. It will then only be necessary

to draw a line from that end of the polygon which belongs to the free end, parallel to the line from that *extreme* end of the load line to the pole; and from the point where it cuts the first centre pier vertical draw the usual line K′ L′ parallel to the original closing line I′ N′. See the dotted lines of this diagram. Such a construction becomes necessary, if a bolt is not employed, as soon as N′ L′, located by the formula of the last section, determines a tension at N, or, in short, when it falls below what would be a tangent at the point N′ to a curve drawn through the vertices of the moment polygon.

165. **Remarks.** — In case the ends of the unloaded draw are lifted by known reactions, the value of **C** is readily ascertained, and the investigation will then proceed as before. Some draws have been designed with the intention that a certain portion of the weight on the centre pier should be carried by the pivot, and the remainder by the wheel circle. Although it is doubtful whether the desired adjustment, if obtained at first, is permanent, the last two cases, of a three-span draw and a draw with tipper, can be combined. The safer way will be to provide for the maximum stresses under either contingency, with the probability that the wheel circle will finally carry the greater part of the weight. The moment diagrams of the two cases can be readily superimposed, as the closing lines for any particular load coincide at T′, Fig. 63.

A careful analytical investigation of draw-spans may be found in "Continuous, Revolving Drawbridges," by Clemens Herschel, Boston, 1875, and an excellent paper on "Turn-Tables for Draws," by C. Shaler Smith, in Transactions Amer. Soc. of C. E., August, 1874, vol. iii., No. 4, xcii. It is probably unnecessary to amplify further, or to contrive other methods of support or construction. The flexibility of this method of area moments has been shown, and any one who understands what precedes will have no difficulty in making the application to any special modification. The diagrams will resemble in most particulars those already given in Fig. 61, slightly modified by the double pier moments.

APPENDIX.

Maximum Stresses from Concentrated Moving Loads.—The figure, Plate X, represents a bridge of 120 feet span, weighing 36 tons for one truss, with a live load of wheel weights, as marked on the line *a x*, at distances noted between the wheels.

Mark the relative positions of the wheel loads on the horizontal line *A B*, to a scale of feet, placing the first load or wheel wherever convenient, and allowing the train to extend indefinitely beyond *B*. Drop verticals from the loaded points; lay off on the vertical line *a x* the wheel loads in order, to a scale of tons or pounds; assume a pole *o*, and proceed to draw the equilibrium polygon A' H' K' for this live load only. The portion in advance of the engine will be a straight line.

Now place the truss *C K* above the train so that the first panel-joint *D* from the abutment *C* shall be vertically over the forward driver. Drop verticals from all the panel-joints to cut the equilibrium polygon, as shown by the full lines, and space off several more panel distances by similar lines at each end of the truss.

The closing line $C'K'$, vertically below *C K*, will cut off ordinates proportional to the bending moments at all joints for the given live load in its given position. As it is easier in the drawing to move the span *C K* than the live load and its accompanying equilibrium polygon, imagine the span moved one panel to the left, when $A'J'$ will be the closing line, and the forward driver will rest at *E*. The other closing lines shown in the figure may now be added, all having a constant horizontal projection equal to the span, and cutting off ordinates proportional to bending moments at panel-joints, as the load traverses the bridge.

If the curve 1—1 is sketched to cut these closing lines at points

distant one panel horizontally from their left extremities, the vertical ordinates between the curve 1—1 and the equilibrium polygon will give all the bending moments at joint D as the train passes, and it is easy to select the maximum value. The maximum ordinate comes under a weight, and may be found quickly by scale, or by placing the edge of a drafting triangle on that part of 1—1 which is apparently straight, and then sliding it down until tangent to the equilibrium polygon; the desired ordinate traverses that point. Thus we find the ordinate D', marked with a heavy line, and showing that the maximum moment at D occurs when the forward driver is on that point.

The curve 2—2 cuts the closing lines at points distant horizontally *two* panels from their left extremities, and the maximum ordinate is found at E', showing for the given loads that the maximum moment at E occurs when the middle driver is upon that point. The curves 3—3 and 4—4 are drawn for points on the closing lines *three* and *four* panel lengths from the end, and the ordinates F' and G' are obtained; hence when the rear driver is on the joint F, and again the second wheel under the tender on the joint G, the train will produce maximum moments at these joints. The greatest chord stress, by ordinate G' for the middle joint, is 50 tons when the head of the engine is between D and E, while the position of train shown in the sketch, with closing line $C'K'$, gives only 45.15 tons for middle ordinate and chord-stress.

The weight of the bridge is more easily treated by itself, and the resulting stresses are then added to those arising from live load, although dead and live load may be combined at once.

The bending moments or chord-stresses from dead load are given by the usual parabola below, and can be added to the ones just obtained. The chord-stress diagram for live load can be advantageously drawn to a larger scale on a drawing board. As the pole distance was assumed at twenty tons and the height of the truss twenty feet, the ordinates D', E', etc., give chord-stresses in this case.

Next, to find the maximum shears in the successive panels:— From o draw $o\ k$, $o\ j$, $o\ i$, $o\ h$ and $o\ g$ parallel to the respective closing lines of the equilibrium polygon which start from K', J', I', etc.; and from the points of division between the first few weights

on the load line at a and below, draw horizontal lines to the respective verticals, so as to obtain the broken line $a\ b\ m\ n\ l\ d$, the usual line in a shear diagram for concentrated loads. When $C'\ K'$ is the closing line, at which time the leading driver is at D, $k\ a$ will be the reaction at C, and any ordinate from the horizontal line k—1′ to the broken line will be the *resultant* shear at the corresponding point in the bridge. As the load on $C\ D$ is carried by the stringers to the points C and D, we desire to find the portion which affects the brace, in addition to the shear from the loads at D and all points to the right of D. The two wheels of the engine truck rest in this panel. Draw the ordinate 1′—s under the centre of gravity of these two weights ; as they are equal, the ordinate will come halfway between them. Project b horizontally to e, where it strikes the vertical through D ; and l to c, where it strikes the vertical through C ; join $e\ c$. Then will $s\ f$ be the portion of the weight $b\ m + n\ l$ carried at D ; for

$$s f : c f = e d : c d, \quad \text{or } s f = \frac{e\,d \,.\, c f}{c\,d} = \frac{(b\,m + n\,l)\ c f}{C\,D},$$

and $(1'—f) + f\,s$ or 1′—s will be the maximum shear in the first brace from rolling load.

If one does not feel sure that the maximum shear in that brace exists when the forward driver rests at D, the load may be advanced or withdrawn a little, by moving the closing line $C'K'$ in the reverse direction, finding the new point k and horizontal line k—1′, moving the verticals under the panel-joints C and D correspondingly, and drawing $c\ e$ in its new position. If the forward driver is brought into the first panel, the vertical projection of $c\ e$ must be increased, to cover all the load in the panel. In either case, it will be found that more is subtracted from one end of 1′—s than is added to the other, so that the present construction gives the desired maximum shear from live load. The two changes, for movement of load two feet either way, are shown by dotted lines.

Similarly j—2′ will cut off the maximum shear 2′—s in the second panel, and 3′—s, 4′—s and 5′—s will be the desired shears in the third, fourth and fifth panels. A line from s, parallel to the braces, will give the stresses in those members, measurements being taken to the points on it cut by the horizontals through 1′, 2′, 3′, etc.

The shear diagram for dead load is given by the usual triangle,

and in this case it is evident that the shear from dead load will overpower that from live load in the sixth panel, so that no ordinate above 5′—*s* is needed.

Bending Moments on Pins.—If the forces which act *at any one time,* in the pieces assembled upon a pin at a given joint, are decomposed into horizontal and vertical, or rectangular components, one set of components may be imagined revolved through a right angle, and they may then be plotted on two load lines with the same *H,* and moment polygons may be drawn on the two sides of a horizontal line which represents the pin. Then can the resultant moment at any section, and the maximum moment, be found by constructing a right-angled triangle, whose legs are the two moments at the section; the hypothenuse, when multiplied by *H,* will measure the desired moment. The best arrangement of pieces on the pin, and the utility of introducing a middle bearing, if the moment would otherwise be large, can then be studied.

A convenient rule, to be followed as much as possible, is to place those pieces in juxtaposition whose stresses tend to balance one another. Thus a diagonal should come between the vertical and the chord piece which resists its stress. Chord bars in adjacent panels should alternate on the pin.

Plate I

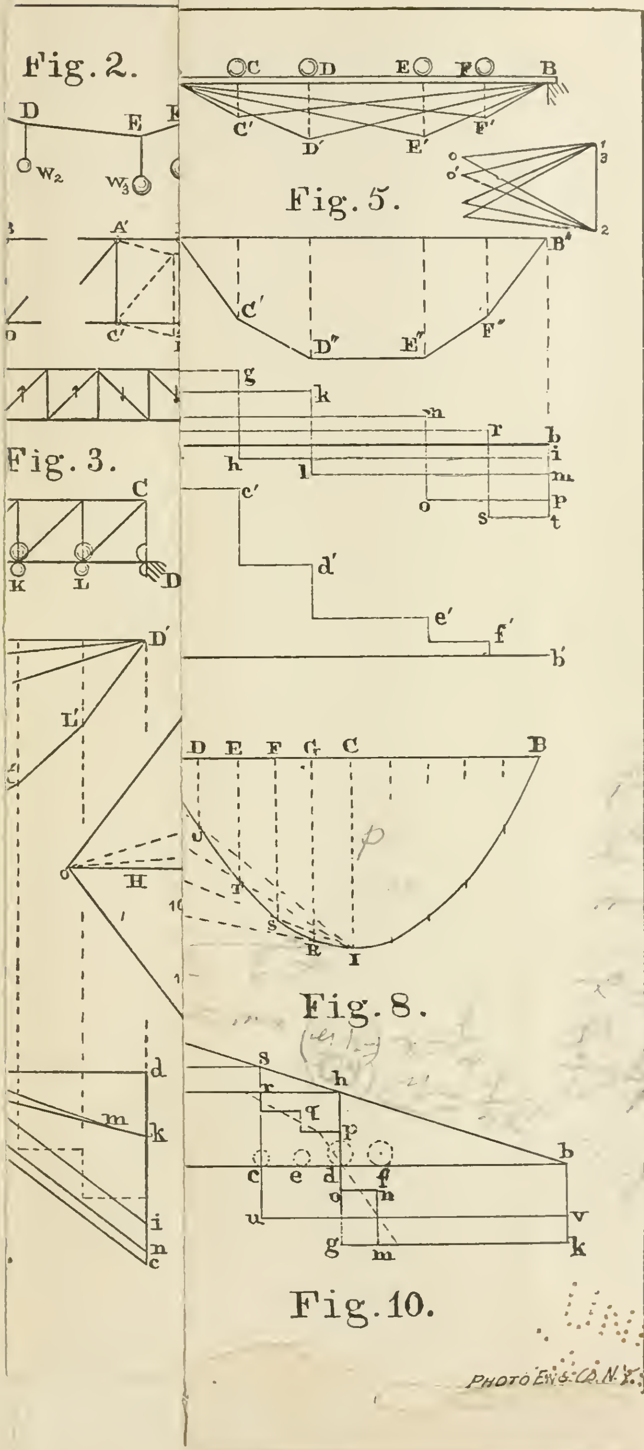


Fig. 2.

Fig. 3.

Fig. 5.

Fig. 8.

Fig. 10.

PHOTO ENG. CO. N.Y.

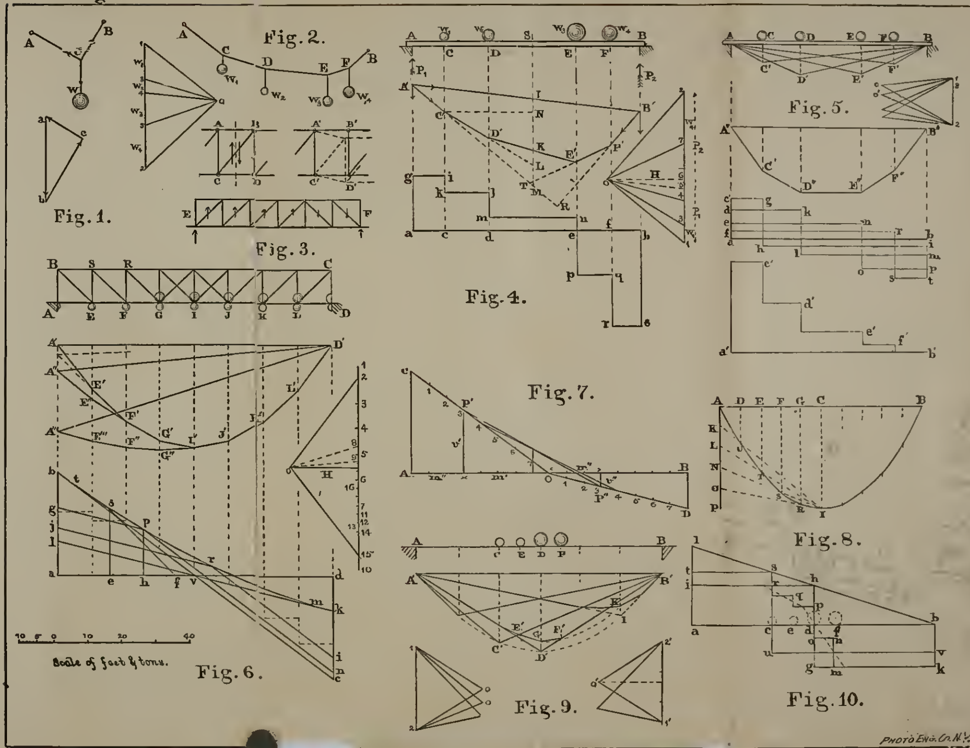
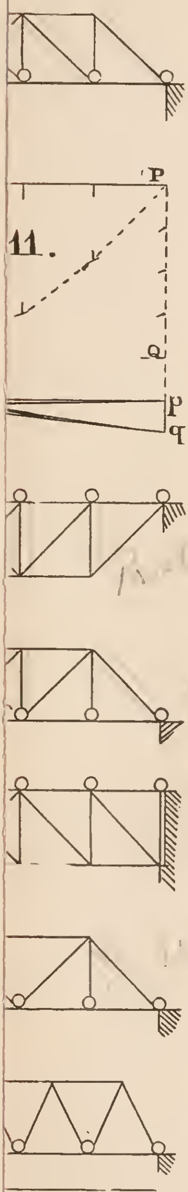


Fig. 1.
Fig. 2.
Fig. 3.
Fig. 4.
Fig. 5.
Fig. 6.
Fig. 7.
Fig. 8.
Fig. 9.
Fig. 10.
10 5 0 10 20 40
Scale of feet & tons.
PHOTO ENG. CO. N.Y.

Plate II.



11.

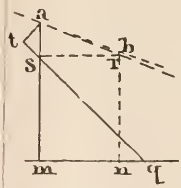


Fig. 13.

Fig. 16.

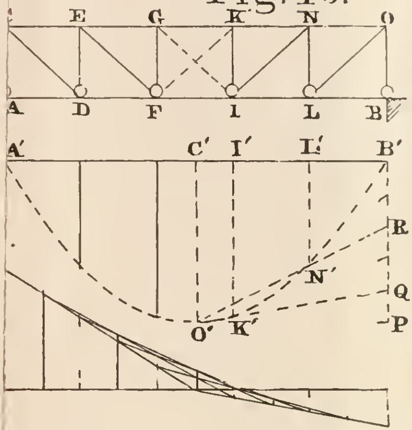
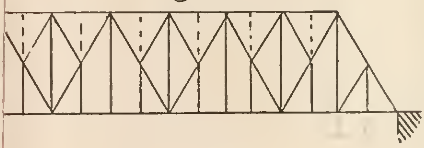
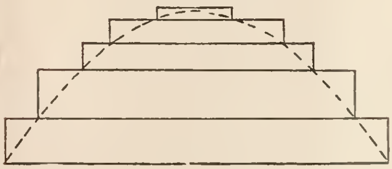


Fig. 18.



19.

Fig. 14



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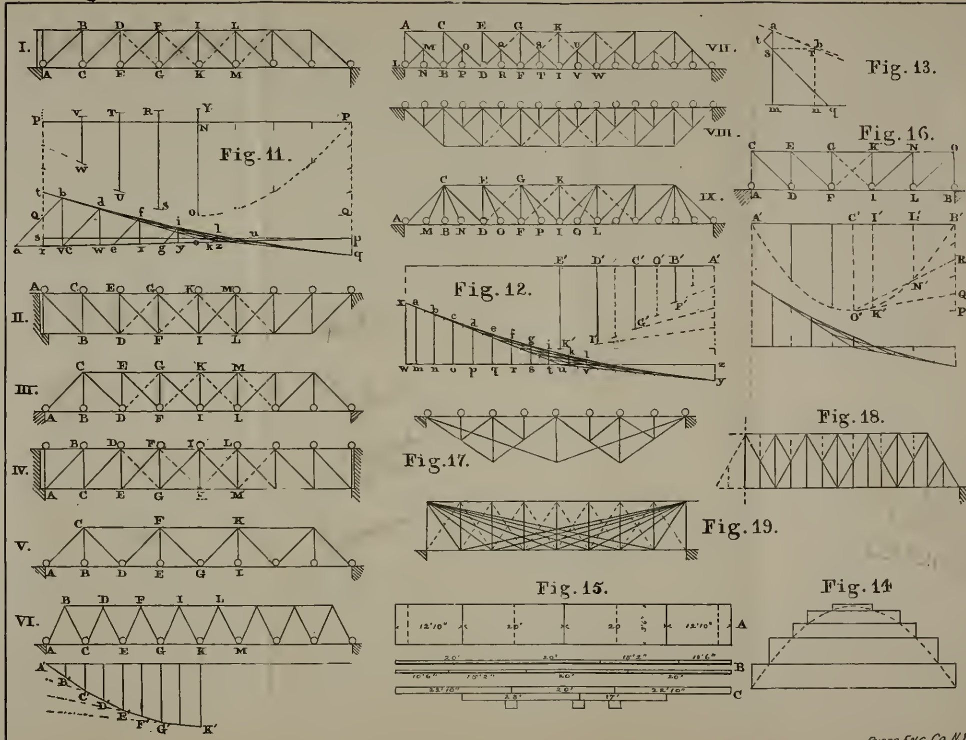
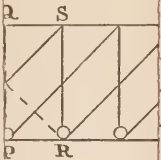

I.
II.
III.
IV.
V.
VI.
VII.
VIII.
IX.
Fig. 11.
Fig. 12.
Fig. 13.
Fig. 14
Fig. 15.
Fig. 16.
Fig. 17.
Fig. 18.
Fig. 19.
12'10"
20'
20
12'10"
A
B
C
PHOTO ENG Co. N.Y.



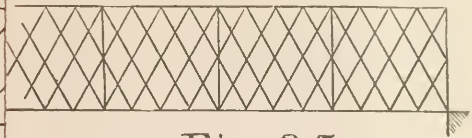


Fig. 25.

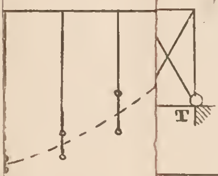


Fig. 24.

Fig. 26.

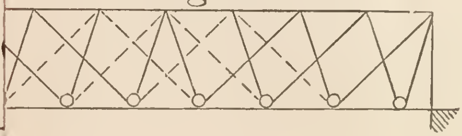
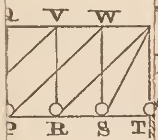
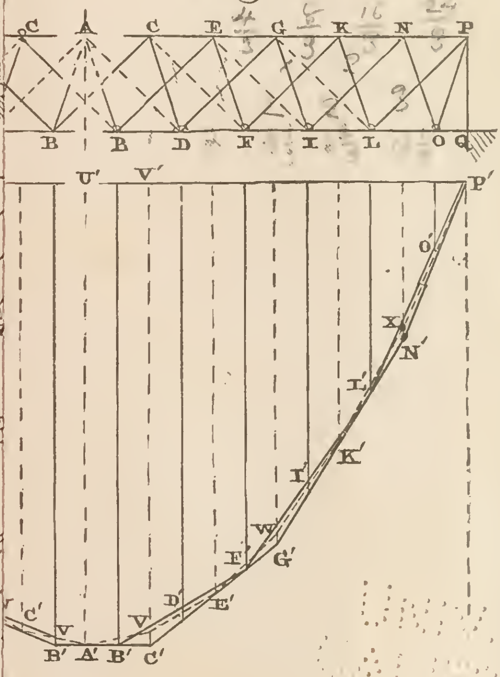


Fig.2

9.

Fig. 28.





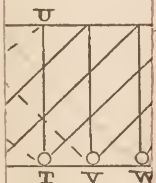
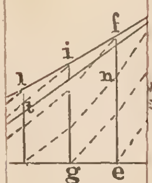


Fig.23



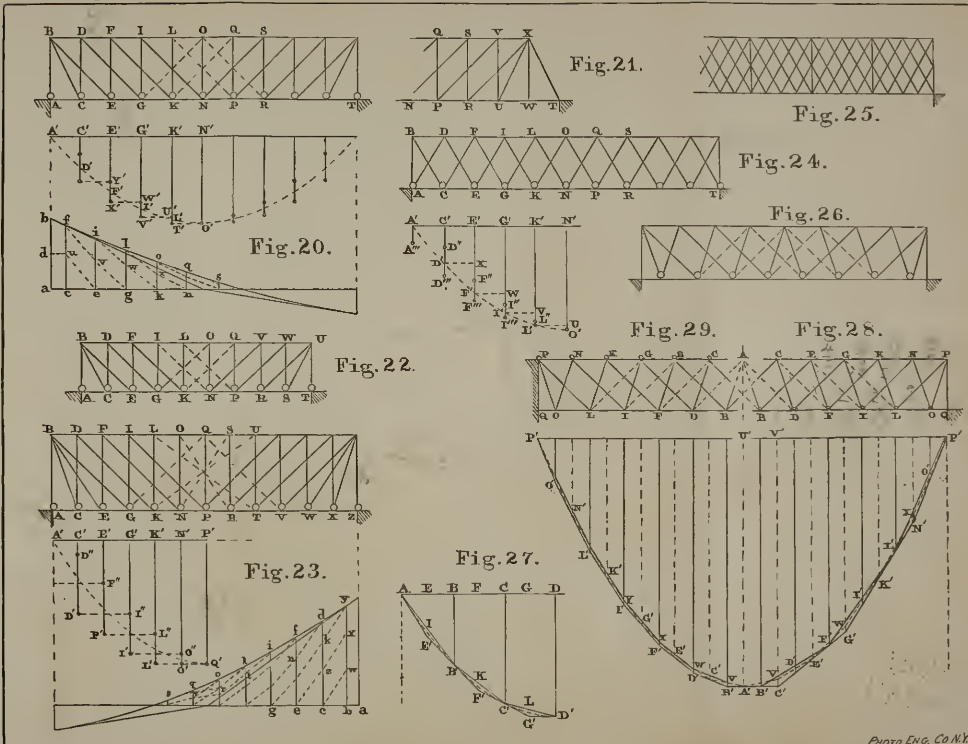


Fig. 20.
Fig. 21.
Fig. 22.
Fig. 23.
Fig. 24.
Fig. 25.
Fig. 26.
Fig. 27.
Fig. 28.
Fig. 29.
PHOTO ENG. CO N.Y.

Plate IV.

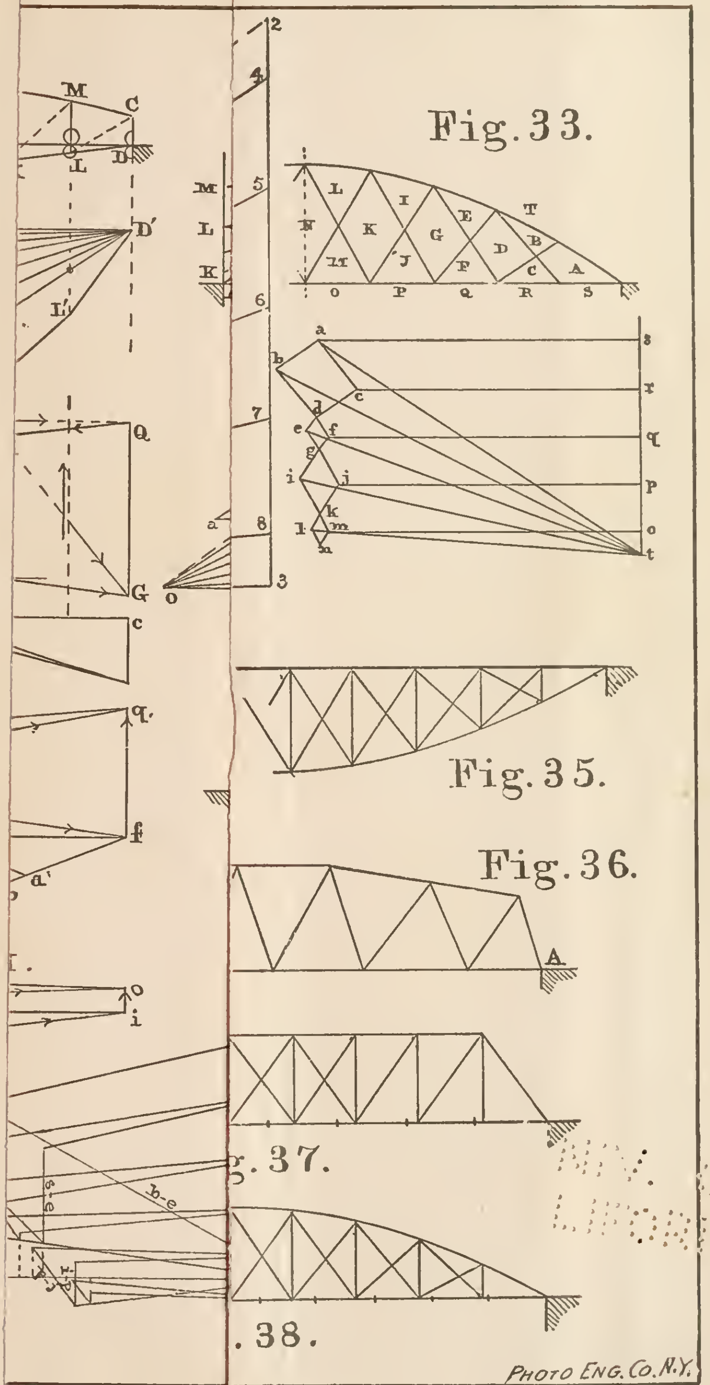


Fig. 33.

Fig. 35.

Fig. 36.

g. 37.

. 38.

PHOTO ENG. CO. N.Y.

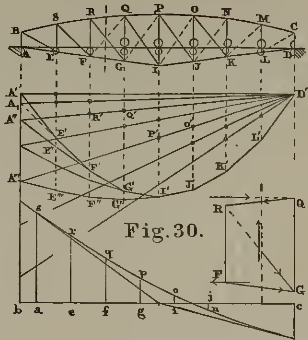


Fig. 30.

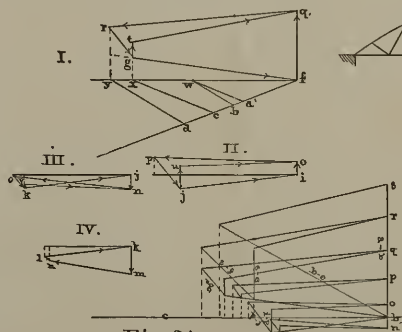


Fig. 31.

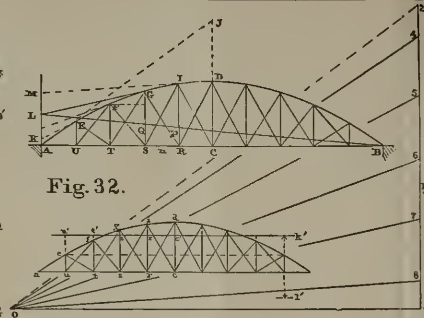


Fig. 32.

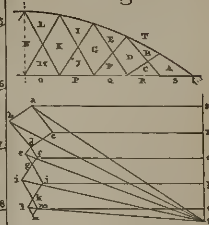


Fig. 33.

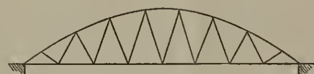


Fig. 34.

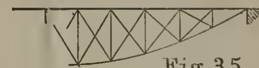


Fig. 35.

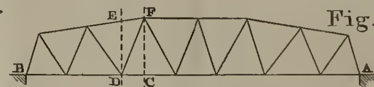


Fig. 36.

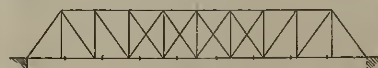


Fig. 37.

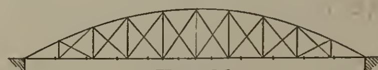
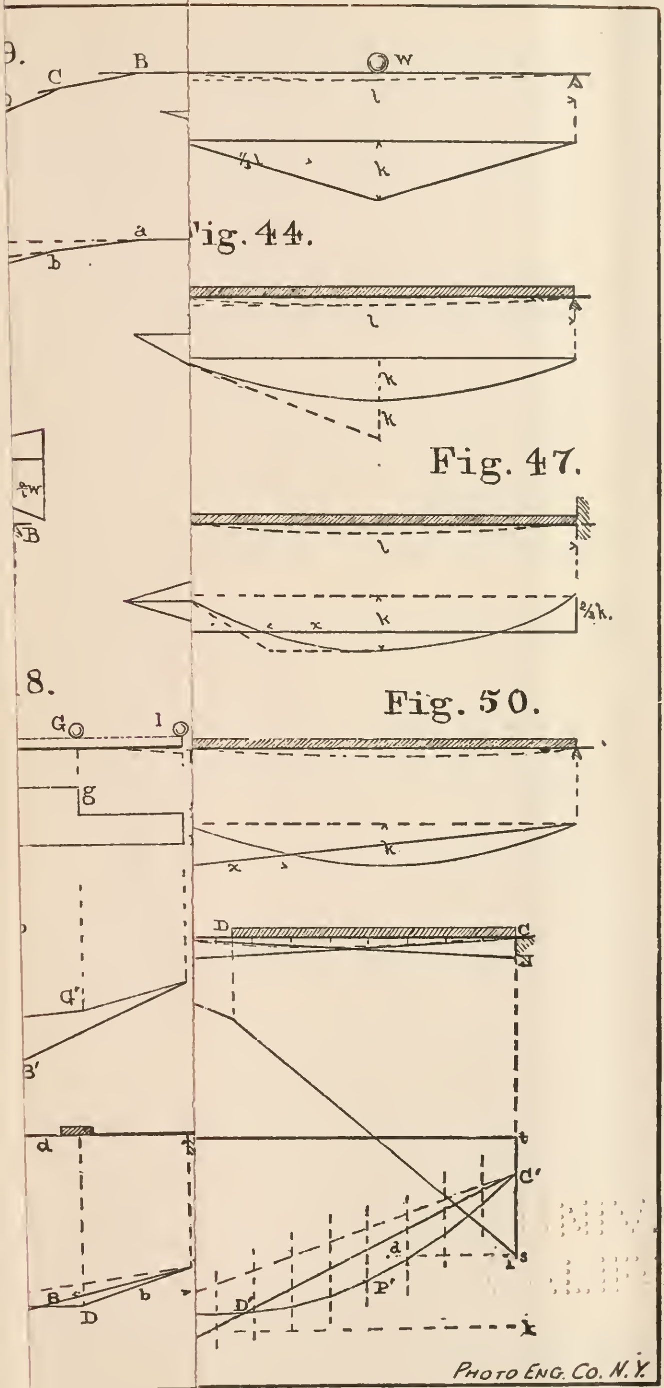


Fig. 38.

Plate V.



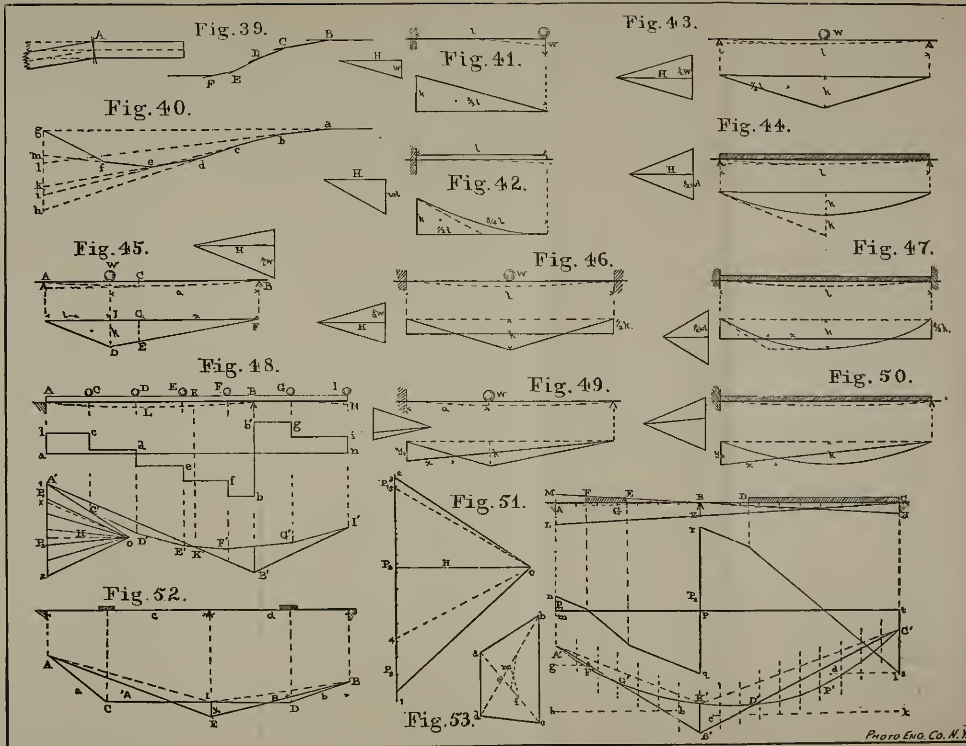


Fig. 39. Fig. 40. Fig. 41. Fig. 42. Fig. 43. Fig. 44. Fig. 45. Fig. 46. Fig. 47. Fig. 48. Fig. 49. Fig. 50. Fig. 51. Fig. 52. Fig. 53.

Plate VI.

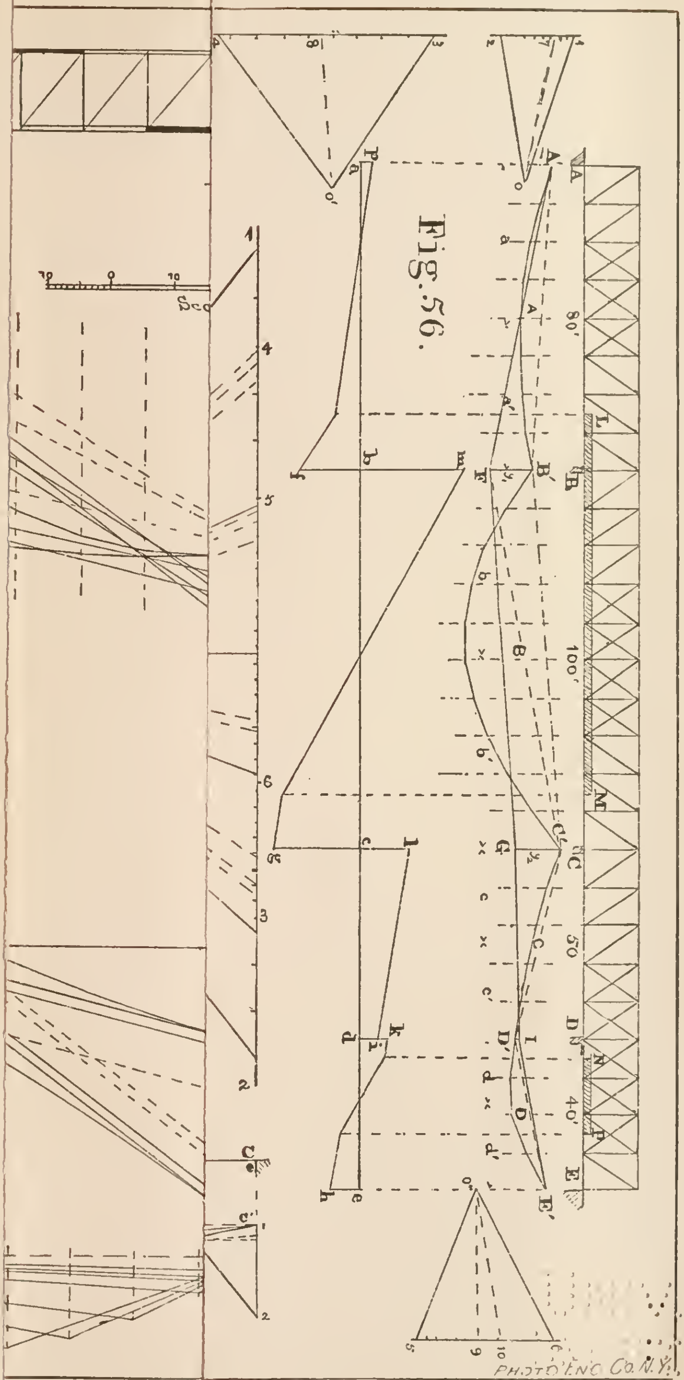


Fig. 56.

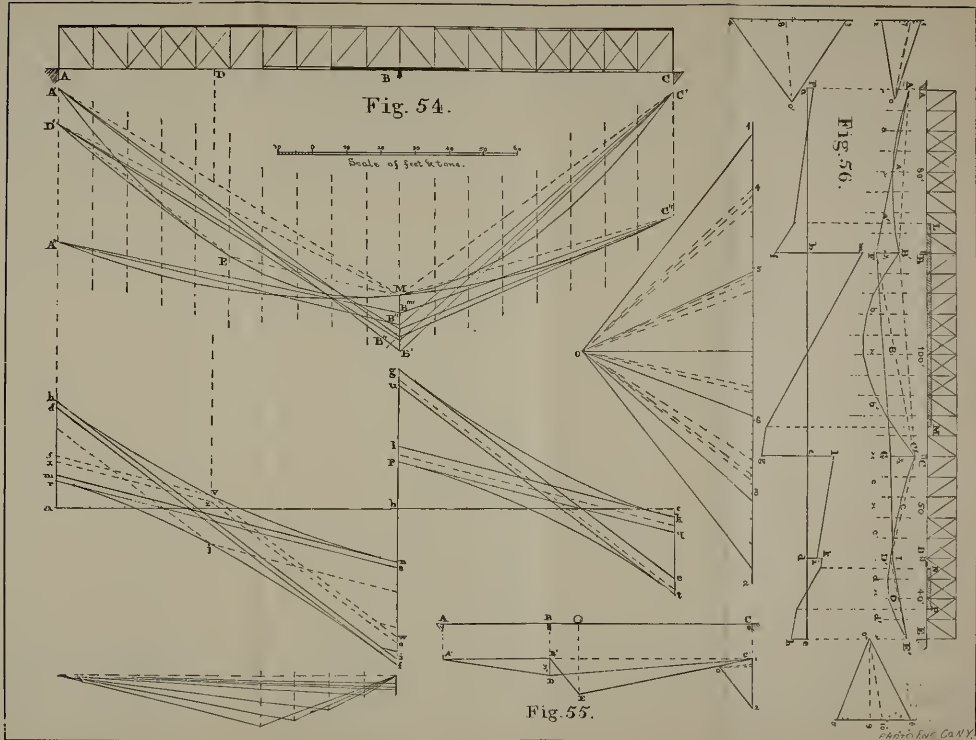


Fig. 54.
Scale of feet & tons.
Fig. 55.
Fig. 56.
PHOTO ENG. CO. N.Y.

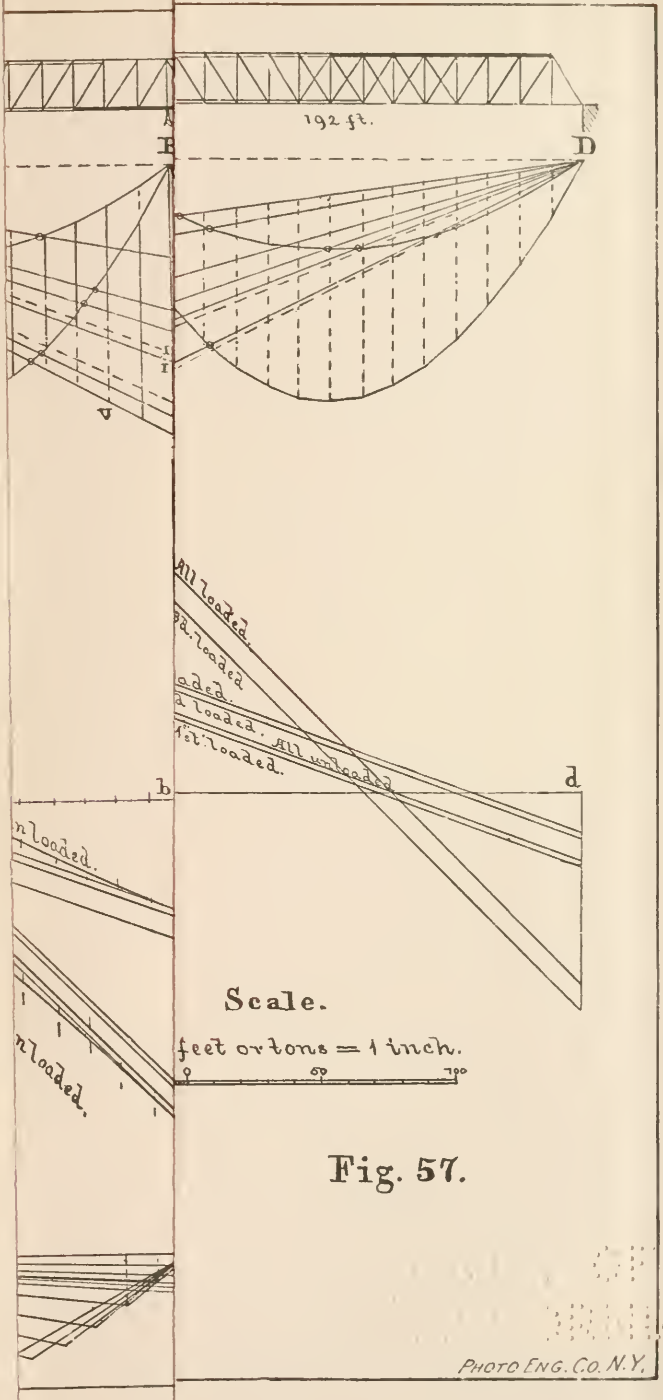
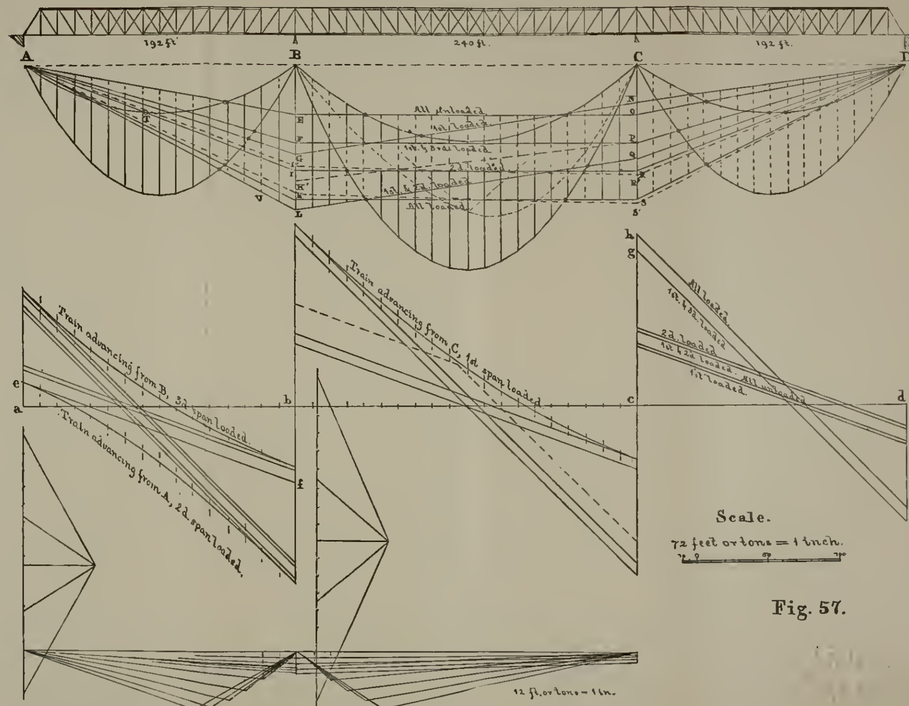

192 ft.
A
B
D
v
All loaded.
loaded.
loaded.
loaded.
1st loaded.
All unloaded.
b
d
nloaded.
nloaded.
Scale.
feet or tons = 1 inch.
0
50
100
PHOTO ENG. CO. N.Y.

Fig. 57.



Scale.

72 feet or tons = 1 inch.

Fig. 57.

PHOTO ENG. CO. N.Y.

Plate VIII.

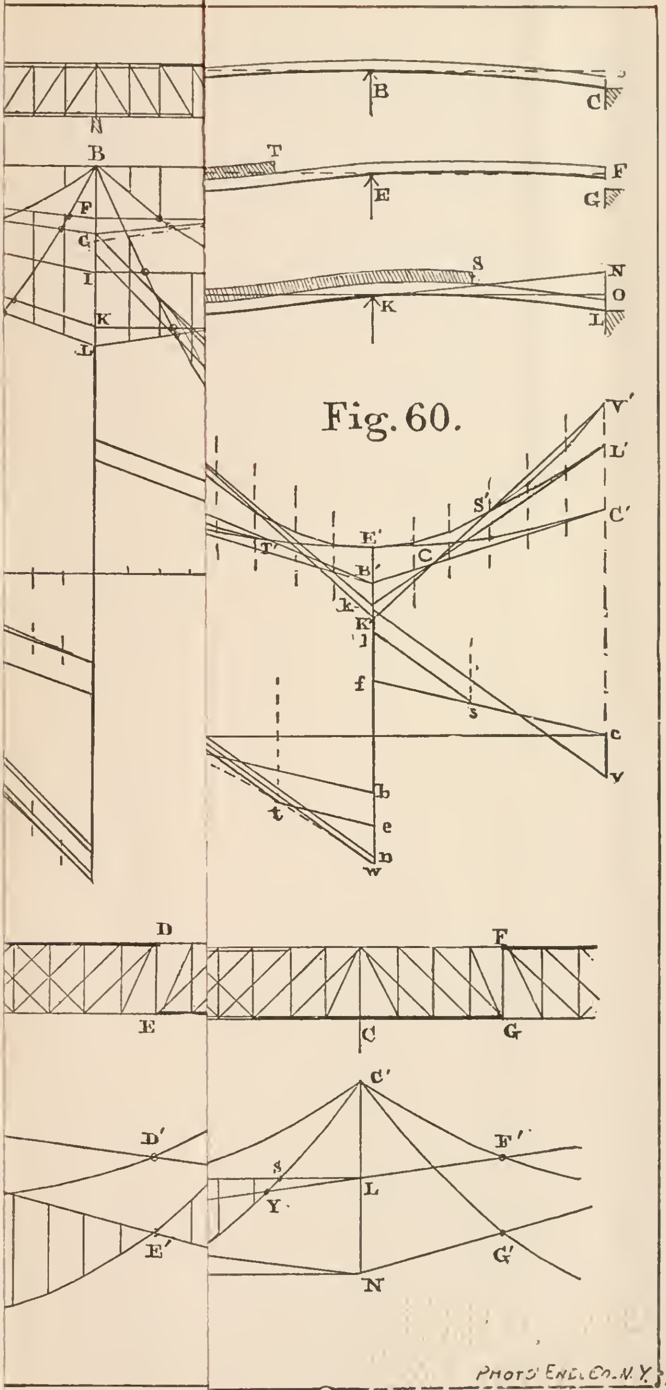


Fig. 60.

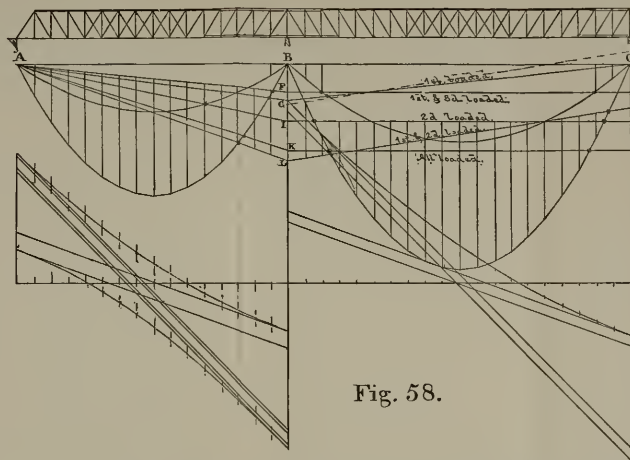


Fig. 58.

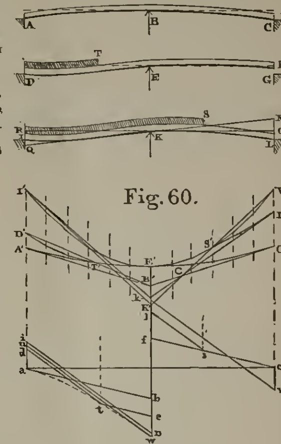


Fig. 60.

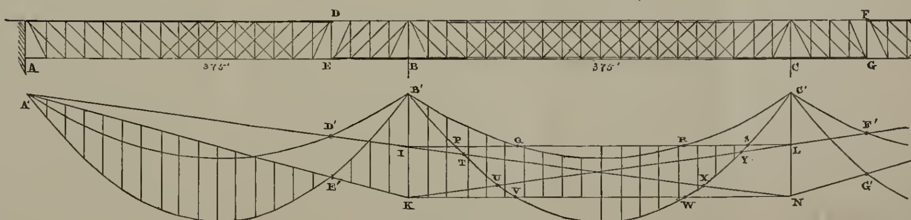
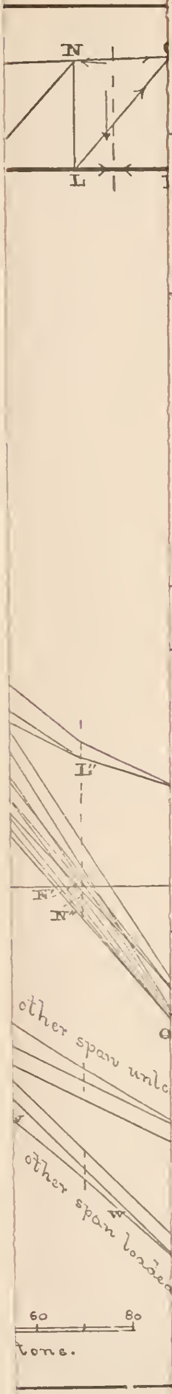


Fig. 59.

PHOTO ENG. CO. N.Y.

Plate IX.



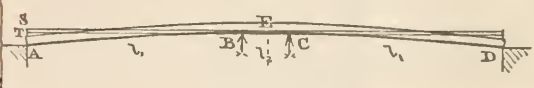
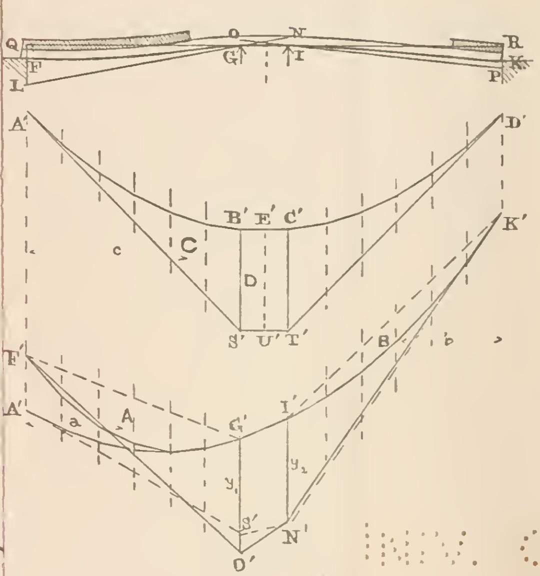


Fig. 62.



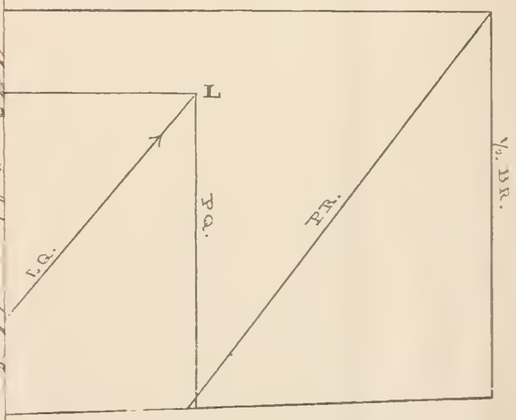


PHOTO ENG. CO. N.Y.

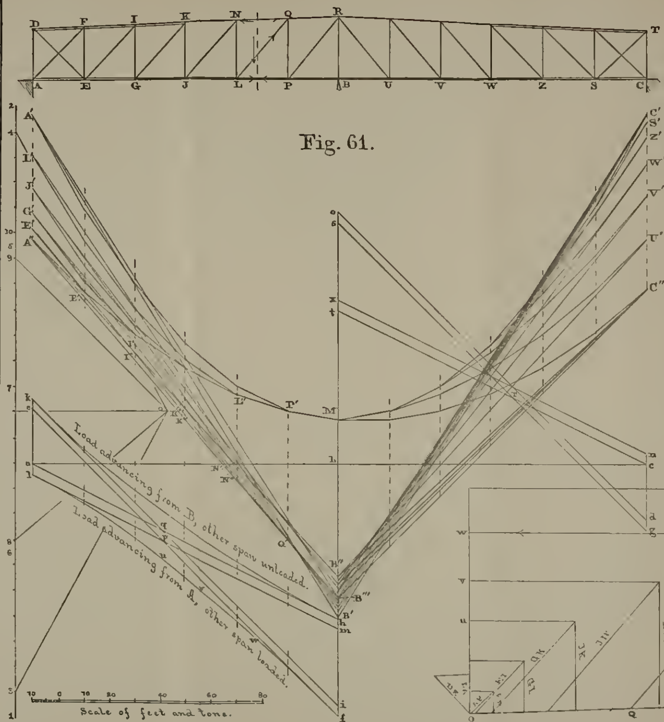


Fig. 61.
Load advancing from B, other span unloaded.
Load advancing from A, other span loaded.
Scale of feet and tons.

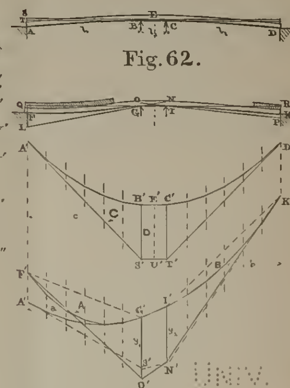


Fig. 62.

ncentrated Loads—General Solution.

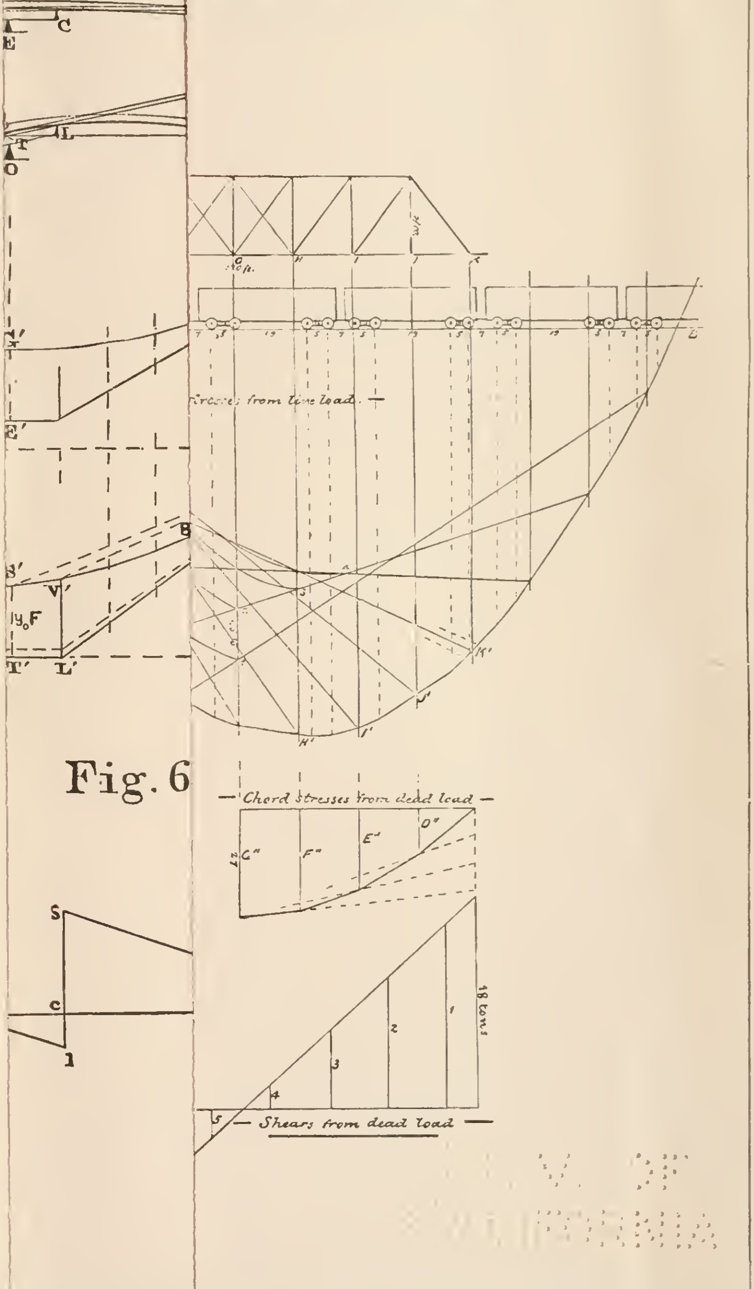


Fig. 6

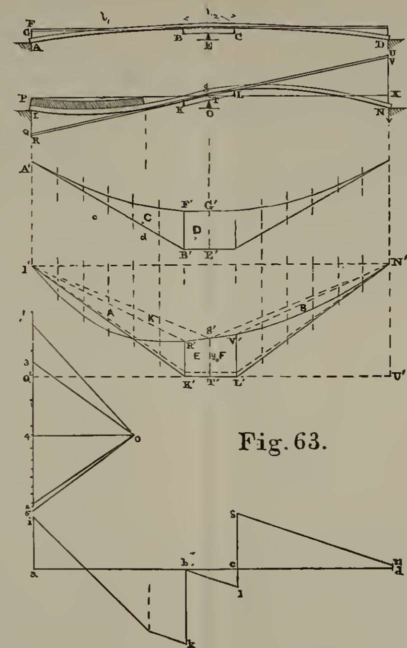


Fig. 63.

Maximum Stresses from Concentrated Loads—General Solution.